Applications of (discrete) quantum walks

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IC1405 COST meeting
Lisbon, 24th Feb 2016

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Motivation

Quantum computation aims at harnessing the properties of quantum systems in order to store, process, and transmit information. However, we do not know how to utilize the advantages of quantum systems.

Quantum walks are useful for

- porting classical concepts, especially related to graph exploration, to the quantum domain;
- modelling quantum networks (ie. systems of connected processing units);
- studying non-classical (and non-quantum) properties of physical teories.

Quantum encoding of information (0-th quantization)

We start with the basic requirement for the pure states to be elements of a linear space.

Rule (0-th quantization of pure states)

Let us map the states as $0 \mapsto \binom{1}{0}$ *and* $1 \mapsto \binom{0}{1}$.

For a bit, the states form a set $\{0, 1\}$, and if one aims to add two elements from this set, the result, *ie.* $a0 + b1$, is not a valid state.

Fact (Dirac notation)

$\binom{1}{0} \equiv |0\rangle$, $\binom{0}{1} \equiv |1\rangle$.

Quantum encoding of information (0-th quantization)

As $|0\rangle$ and $|1\rangle$ are just plain vectors, the state of the quantum bit can be represented by any combination of the form

$$x_0|0\rangle + x_1|1\rangle, \quad x_0, x_1 \in \mathbb{C}.$$

Quantum processing of information (0-th quantization)

In the classical case we have only two allowed reversible operations on one bit: identity and negation.

Rule (0-th quantization of one-bit circuits)

For a given reversible classical operation, its quantum counterpart is defined by its application on the base states $|0\rangle$ *and* $|1\rangle$.

For operations acting on one bit, the simplest example is the binary negation (NOT) operation. Its quantum counterpart is given by a matrix which simply interchanges the base states,

$$NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Quantum processing of information (0-th quantization)

The simplest example of a non-classical gate is the $\sqrt{NOT}$ gate

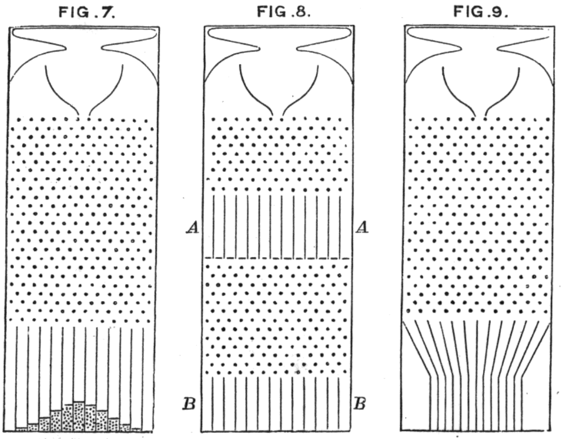
$$\sqrt{NOT} = \frac{1}{2}\begin{pmatrix} i+1 & i-1 \\ i-1 & i+1 \end{pmatrix},$$

which results in states "half way" between the input state and the negated input state.

$$\sqrt{NOT}|0\rangle = \frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle,$$
$$\sqrt{NOT}|0\rangle = \frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle.$$

From random walks to quantum walks

Simple model of random walk



(F. Galton, Natural Inheritance, Macmillan, 1889, p. 63. See: http://galton.org/)

From random walks to quantum walks

Reversible evolution

Quantum evolution must be linear and must preserve probabilities. This is reflected in the fact that quantum gates are unitary and thus reversible.

The direct counterpart of classical evolution of the Galton board encoded in quantum states reads

$$|x-1\rangle \mapsto \alpha|x-2\rangle + \beta|x\rangle$$
$$|x+1\rangle \mapsto \alpha|x\rangle + \beta|x+2\rangle$$

The initial states are orthogonal, $\langle x-1|x+1\rangle = 0$.

Quantum coins

One needs a coin in order to fully specify quantum evolution.

From random walks to quantum walks
Reversible evolution

The basic model of quantum walk is described as

$$U = S(C \otimes \mathbb{I})$$

where S (shift) describes a step of the walker, and C (coin) governs the choice of direction. For grapf of degree 2, shift is defined as a controlled operation of the form

$$\sum_x |x-1\rangle\langle x| \otimes |0\rangle\langle 0| + |x+1\rangle\langle x| \otimes |1\rangle\langle 1|.$$

From random walks to quantum walks

Quantum walks vs. quantum random walks

Decoding quantum information

As the final step of each quantum algorithm one has to perform a measurement. This is required to decode final information and make it useful for the clasical world.

Quantum walks vs. quantum random walks

In quantum walks the measurement is performed only at the end of evolution. In quantum **random** walks, the measurement is performed after each step.

Quantum walks...

... are not random – the state is completely described at each time step.

From random walks to quantum walks

Spreading speed

For quantum walks we get *faster* spreading of the probability distribution.

- random walk – standard deviation grows as $\sigma \sim \sqrt{t}$
- quantum walk – standard deviation grows as $\sigma \sim t$

From random walks to quantum walks

Limiting distribution

The probability of finding a particle at position v after n steps is obtained after averaging over the coin

$$p(v,n) = \sum_{c} |\langle c, v|\phi_n\rangle|^2,$$

However, $p(v,n)$ is quasi-periodic and it was suggested to consider time-averaged limiting distribution

$$\bar{p}(n,t) = \frac{1}{t}\sum_{s=1}^{t} p(n,s). \quad (1)$$

which converges with $t \to \infty$ to the limiting distribution $p(v)$.

Search algorithms and quantum walks

In the standard quantum walk, the coin operator is constant. To find some element, we have to mark it with the coin operator using information about the current vertex,

$$-\mathbb{I} \otimes |v_0\rangle\langle v_0| + G \otimes (\mathbb{I} - |v_0\rangle\langle v_0|).$$

Such position-dependent coin can be used to *mark* the target vertices.

Search algorithms and quantum walk

Examples of application of quantum walks for searching problems include

- element k-distinctness [Ambainis 2003 and 2007]
- subset finding [Child and Eisenberg 2009]
- triangle finding [Magniez, Santha, Szegedy 2005]

Search algorithms and quantum walks

k-distinctness

We have a set $\mathcal{X}$ with N elements combined with some values from a finite set V, assigned by function $f : \mathcal{X} \mapsto V$. The algorithm should return a subset $\{x_1, x_1, \ldots, x_k\}$, such that $X_1, f(x_1, \ldots, (x_k, f(x_k)\}$ belong to some given subset of $(\mathcal{X} \times V)^k$.

This problem can be represented a s bipartite graph. Quantum walk defined by this graph can output the tuple using $O(N^{\frac{k}{k+1}})$ queries to the oracle. Classically $O(N)$ queries are required.

Quantum Magnus-Derek game

Magnus-Derek game is played by two players: Derek (from *direction*) and Magnus (from *magnitude*), who operate by moving a token on a round table (cycle) with n nodes $0, 1, \ldots, n-1$.

The cardinality of the set of positions visited by the token when both players play optimally is well defined and reads

$$f^{\star}(n) = \begin{cases} n & \text{for } n = 2^k, \\ \frac{(p-1)n}{p} & \text{for } n = pm, \end{cases}$$

with p being the smallest odd prime factor of n.
By $r(n)$ we denote the number of moves required to visit the optimal number of nodes.

Quantum Magnus-Derek game

In the quantum scenario the position of the token is encoded in a state $|x\rangle \in \mathbb{C}^n$.

The evolution is governed by the operator

$$S = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n} |m,0\rangle\langle m,0| \otimes |k+m\rangle\langle k| + \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n} |m,1\rangle\langle m,1| \otimes |k-m\rangle\langle k|,$$

Quantum Magnus-Derek game

Definition

We say that the position x is visited in t steps, if for some step $i \leq t$ the probability of measuring the position register in the state $|x\rangle$ is 1.

Attained position

We say that the position x is attained in t steps, if $|x\rangle$-measured exploration walk has a $(t, 1)$ concurrent $(|\psi_0\rangle, |x\rangle)$ hitting time, i.e. the exploration walk with initial state $|\psi_0\rangle$ has a probability of stopping at a time $t < T$ equal to 1.

Quantum Magnus-Derek game

The moves performed by Magnus allow him the sampling of the space of positions using $r(n) = n - 1$ steps and Derek is not able to prevent Magnus from attaining all nodes using $r(n)$ moves.

On the other hand, Derek is able to prevent Magnus from visiting all nodes. He can achieve this by applying the strategy given as follows.
For steps $i = 1, \ldots, n$ perform the following gate

$$D_i = \begin{cases} H & \text{if } i \text{ is odd} \\ \mathbb{I} & \text{if } i \text{ is even} \end{cases},$$

where H denotes the Hadamard gate.

JAM, P. Sadowski, arXiv:1308.5923

Quantum walks with higher-dimensional coins

Let us denote by $|\psi\rangle = |c\rangle \otimes |x\rangle$ the state of the quantum walk with $|c\rangle \in \mathbb{C}^3$ and $|x\rangle \in \mathbb{C}^n$.

Lively quantum walk on a cycle

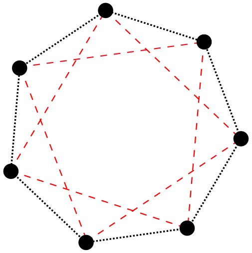
Lively quantum walk on an n dimensional cycle with liveliness $0 \leq a \leq \lfloor \frac{n}{2} \rfloor$, is defined by the shift operator of the form

$$\begin{aligned} S_x^{(n,a)} &= |0\rangle\langle 0| \otimes |x-1 \pmod{n}\rangle\langle x| \\ &+ |1\rangle\langle 1| \otimes |x+1 \pmod{n}\rangle\langle x| \\ &+ |2\rangle\langle 2| \otimes |x+a \pmod{n}\rangle\langle x|. \end{aligned}$$

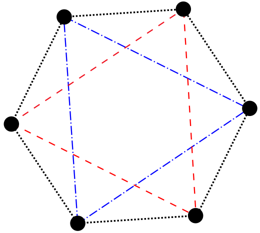
Quantum walks with higher-dimensional coins

Theorem

If $\mathrm{GCD}(a, n) > 1$ *then the limiting time-averaged probability distribution is periodic with period equal to* $\mathrm{GCD}(a, n)$.



(a) Aperiodic case



(b) Periodic case

Quantum walks are good for

- implementing quantum search $\mapsto$ quantum seach algorithm can be described by quantum walk on complete graph,
- modelling of mobile agents in quantum networks $\mapsto$ routing of quantum information based on a quantum state can be described by quantum walks,
- analyzing the influence of the graph structure on the resulting quantum evolution $\mapsto$ quantum walks can be easily applied to parametrize quantum processes using information about the underlying graph.