Exploring quantum networks with faulty sense of direction

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We consider a scenario of exploring a network of quantum information processing nodes with a faulty sense of direction. We provide a model of quantum network exploration based on quantum walk on cycle and mobile agents and we study its properties.

Introduction

Among the interesting problems arising in the area of quantum internetworking protocols is the development of methods which can be used to detect errors occurring in large-scale quantum networks. A natural approach for developing such methods is to construct them on the basis of the methods developed for classical networks.

In this work we develop a method for **exploring quantum networks using mobile agents** which operate using information stored in quantum registers. We construct a model of two-person quantum game on a cycle which can be used to analyse the scenario of exploring quantum networks with faulty sense of direction. We analyse the behavior of quantum mobile agents operating using various classes of strategies.

The **Magnus-Derek combinatorial game** has been introduced by Nedev and Muthukrishnan [1] for the purpose of modelling the behavior of mobile agents exploring the ring network with faulty sense of direction. Here the sense of direction refers to the capability of a processor to distinguish between its adjacent communication lines.

The game is played by two players: **Derek** (from ***direction*** or ***distraction***) and **Magnus** (from ***magnitude*** or ***maximization***), who operate by moving a token on a round table (cycle) with n nodes $0, 1, \ldots, n-1$. Initially the token is placed on the position 0. In each round (step) Magnus decides about the number $0 \le m \le \frac{n}{2}$ of positions for the token to move and Derek decides about the direction: clockwise (+ or 0) or counter-clockwise (− or 1). Magnus aims to maximize the number of nodes visited during the game, while Derek aims to minimize this quantity.

Exploring quantum networks

Let us now assume that the players operate by **encoding their positions** on an n-dimensional cycle **in finite dimensional pure quantum states**. To achieve this we introduce a quantum scheme by defining the following quantum version of the game:

1) The state of the system is described by a vector of the form
$$|m\rangle|d\rangle|x\rangle \in \mathbb{C}^{\lfloor n/2 \rfloor} \otimes \mathbb{C}^2 \otimes \mathbb{C}^n.$$

2) The initial state of the system reads $|\psi_0\rangle = |0\ldots 0\rangle$.

3) At each step the players can choose their strategy.

- **3.a)** Magnus operates on his register with any unitary gate $M_i \in \mathbb{SU}(\lfloor n/2 \rfloor)$ resulting in a operation of the form $M_i \otimes \mathbb{1}_2 \otimes \mathbb{1}_n$ performed on the full system.
- **3.b)** Derek operates on his register with any unitary gate $D_i \in \mathbb{SU}(2)$ resulting in a operation of the form $\mathbb{1}_{\lfloor n/2 \rfloor} \otimes D_i \otimes \mathbb{1}_n$ performed on the full system.

4) The change of the token position, resulting from the players' moves, is described by the shift operator

$$S = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n} |m,0\rangle\langle m,0| \otimes |k+m\rangle\langle k| + \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{k=0}^{n} |m,1\rangle\langle m,1| \otimes |k-m\rangle\langle k|,$$

where the addition and the subtraction is in $\mathbb{Z}_n$.

The single move in the game defined according to the above description is given by the operator

$$A_i = S(M_i \otimes D_i \otimes \mathbb{1}_n).$$

We introduce the notion of **visiting** and **attaining** a position to provide the notion of node visiting suitable for analysing quantum superpositions of states.

Definition 1. *We say that the position x is **visited** in t steps, if for some step $i \le t$ the probability of measuring the position register in the state $|x\rangle$ is 1,* i.e.

$$\mathrm{tr}|x\rangle\langle x|(\mathrm{tr}_{M,D}|\psi_i\rangle\langle\psi_i|) = 1.$$

A $|x\rangle$-measured quantum walk from a discrete-time quantum walk starting in a state $|\psi_0\rangle$ is a process defined by iteratively first measuring with the two projectors $\Pi_0 = |x\rangle\langle x|$ and $\Pi_1 = \mathbb{1} - \Pi_0$. If Π_0 is measured the process is stopped, otherwise a step operator is applied and the iteration is continued.

A quantum random walk has a (T, p) concurrent $(|\psi_0\rangle, |x\rangle)$ hitting-time if the $|x\rangle$-measured walk from this walk and initial state $|\phi_0\rangle$ has a probability $\ge p$ of stopping at $t \le T$.

Definition 2. *We say that the position x is **attained** in t steps, if $|x\rangle$-measured exploration walk has a $(t, 1)$ concurrent $(|\psi_0\rangle, |x\rangle)$ hitting time, i.e. the exploration walk with initial state $|\psi_0\rangle$ has a probability of stopping at a time $t < T$ equal to 1.*

With the help of these definitions, one can introduce the concepts of *visiting strategy* and *attaining strategy.*

Definition 3. *If for the given sequence of moves performed by Magnus, there exists t such that each position on the cycle is visited in t steps, then we call such sequence of moves a **visiting strategy**.*

Definition 4. *If for the given sequence of moves performed by Magnus, each position on the cycle is attained, then we call such sequence of moves an **attaining strategy**.*

Application of quantum strategies

Case $n = 2^k$

The optimal strategy for Magnus can be computed at the beginning of the game (see Lemma 2 in [1]). As the moves performed by Magnus allow him the sampling of the space of positions using $r(n) = n - 1$ steps, it can be easily seen that **Derek is not able to prevent Magnus from attaining all nodes** using $r(n)$ moves.

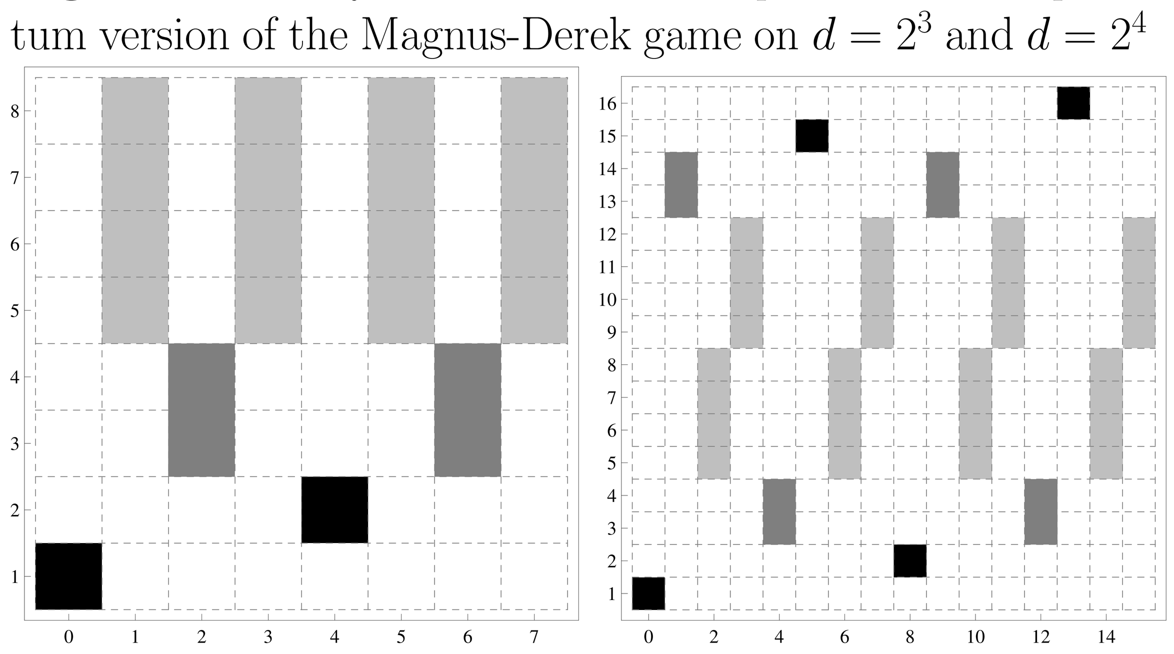
On the other hand, **Derek is able to prevent Magnus from visiting all nodes**. He can achieve this using the strategy given as follows.

Strategy 1. For steps $i = 1, \ldots, n$ perform the following gate

$$D_i = \begin{cases} H & \text{if } i \text{ is odd} \\ \mathbb{1} & \text{if } i \text{ is even} \end{cases}, \qquad (1)$$

where H denotes the Hadamard gate.

Fig. 1 Probability distribution for the position in the quantum version of the Magnus-Derek game on $d = 2^3$ and $d = 2^4$



$n = 2^3$ $n = 2^4$

Case $n = pm$: Position-controlled adaptive strategy

In the situation $n = pm$, without the possibility to perform position-controlled operations Derek can only use classical information about history of choices of Magnus' unitaries. On the other hand, if he is able to decide about his move using the current position, the resulting strategy is more robust.

Let us consider Magnus-Derek game on $n = pm$, $p > 3$, $n \ne 2^k$ positions with p being the least prime divisor of n. When the set of operators available for Derek includes the operators of the form

$$\sum_k \mathbb{1} \otimes D_k \otimes |k\rangle\langle k|$$

where k is an arbitrary position and D_k is an arbitrary local unitary operation then the maximum number of attained positions for Magnus is equal to $n - n/p$ (as in the classical case) and the total number of visited positions is at most 2 (respectively 1 if n is an odd number).

In the simplest case Derek leads to a superposition of two states. In this case he needs only to ensure that the superposition will not vanish – see Fig. 2.

Strategy 2. For any Magnus' strategy based on permutation operators, when $n = pm, p > 3$ and p is a prime number, Derek has to perform the following steps:

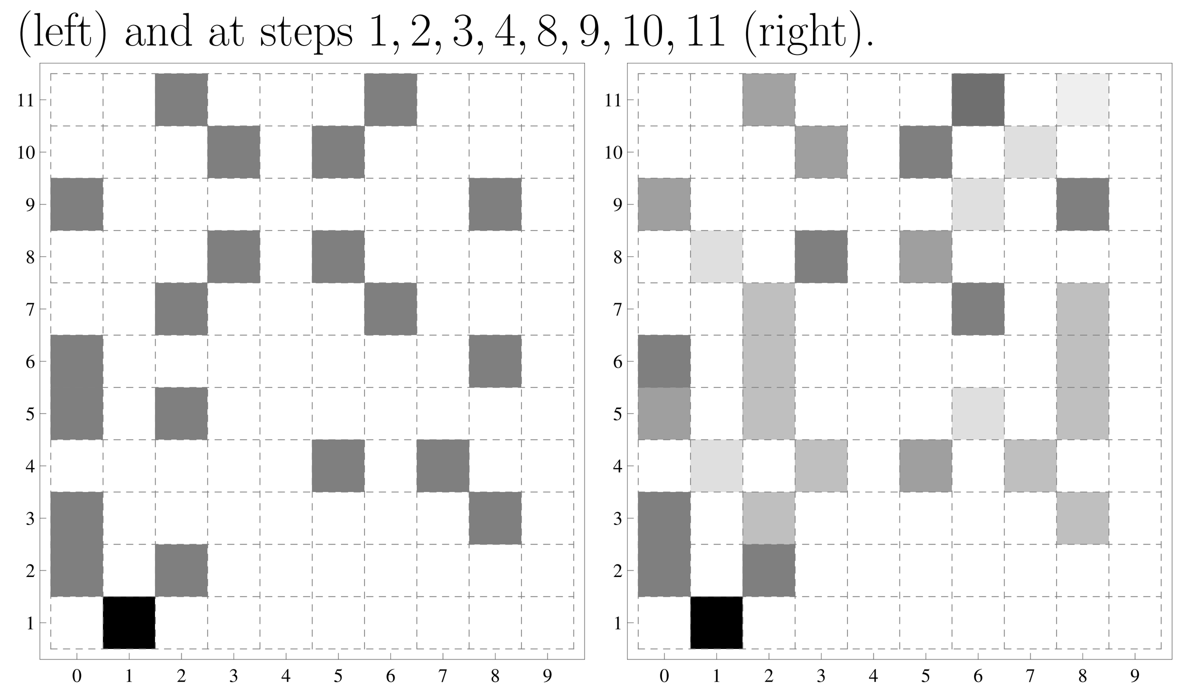
Step 1: Apply the Hadamard gate.

Step 2: If n is even do nothing as long as Magnus move is equal to $n/2$. If n is odd go to **Step 3**.

Step 3: Find a set of $\frac{n}{p} = m$ equally distant positions that is disjoint with already visited positions.

Step 4: Apply classical strategy to both parts of the state using position–controlled operators.

Fig. 2 Strategy 2 with the Hadamard gate at the first step (left) and at steps $1, 2, 3, 4, 8, 9, 10, 11$ (right).



Final remarks

- By extending the space of possible moves, both players can significantly change the parameters of the exploration.
- The modification of a classical strategy that enables both players to preform their tasks efficiently provides an insight into the difficulty of achieving quantum-oriented goals.
- We have shown that without a proper model of adaptiveness, it is not possible for Derek to obtain the results analogous to the classical case.

References

[1] Z. Nedev and S. Muthukrishnan. The Magnus-Derek game, *Theor. Comput. Sci.*, 393(1-3):124 – 132, 2008.

[2] J.A. Miszczak, P. Sadowski, Quantum network exploration with a faulty sense of direction, *Quantum Inf. Comput.*, 14 (13&14), pp. 1238-1250 (2014), arXiv:1308.5923

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