# Computationally feasible bounds for quantum fidelity

Jarosław Miszczak[□,⋆], Zbigniew Puchała[□], Paweł Horodecki[⊗], Armin Uhlmann[⊕], Karol Życzkowski[†]
[□]IITiS PAN, Gliwice, Poland, [⋆]Masaryka University, Brno, Czech Republic, [⊗]Gdańsk University, Gdańsk, Poland,
[⊕]Universität Leipzig, Leipzig, Germany, [†]Jagiellonian University, Kraków, Poland

## Abstract

We present some bounds for fidelity between two quantum states. We discuss two quantities, namely sub-fidelity

$$E(\rho_1,\rho_2) = \mathrm{tr}\rho_1\rho_2 + \sqrt{2}\sqrt{(\mathrm{tr}\rho_2\rho_2)^2 - \mathrm{tr}\rho_1\rho_2\rho_1\rho_2},$$

and super-fidelity

$$G(\rho_1,\rho_2) = \mathrm{tr}\rho_1\rho_2 + \sqrt{(1-\mathrm{tr}\rho_1^2)(1-\mathrm{tr}\rho_2^2)},$$

which give lower and upper bound respectively.

Fidelity between quantum states cannot be directly measured in laboratory. We show that in the case of discussed bounds this is possible and we propose an experimental scheme which can be used to perform this task. Another advantage of $E$ and $G$ is that they can be easily, in contrast to fidelity, computed using standard computer algebra systems.

We discuss to what extends those bounds can be used instead of original fidelity and compare their basic properties. We also present some results concerning the difference between fidelity and presented bounds.

## Introduction

Quantum state is an operator $\rho : \mathcal{H}_N \to \mathcal{H}_N$, which is positive semi-definite ($\rho \geq 0$) and normalised ($\mathrm{tr}\rho = 1$). We denote by $\Omega_N \subset \mathcal{M}_N$ the space of density matrices on $\mathbb{C}^N$.

We define the fidelity between two states as

$$F(\rho_1,\rho_2) = (\mathrm{tr}|\sqrt{\rho_1}\sqrt{\rho_2}|)^2 = ||\rho_1^{1/2}\rho_2^{1/2}||_1^2,$$

where $||\cdot||_1$ is a trace norm, *i.e.* $||A||_1 = \mathrm{tr}|A| = \sum_{i=1}^N \sigma_i(A)$.
In the case of two pure states $\rho_1 = |\phi\rangle\langle\phi|, \rho_2 = |\psi\rangle\langle\psi|$ we have $F(\rho_1,\rho_2) = |\langle\psi|\phi\rangle|^2$.

Fidelity between two diagonal operators is equal to the *Bhattacharyya* coefficient for their eigenvalues.

$$F(\mathrm{diag}(\rho_1), \mathrm{diag}(\rho_2)) = B^2(p,q) = \sum_{i=1}^n \sqrt{p_i q_i}$$

Here $p$ and $q$ are eigenvalues of $\rho_1$ and $\rho_2$ respectively.

We start our analysis of fidelity by expressing it in terms of eigenvalues $\lambda_i,\ i=1,\dots,N$ of the (positive) matrix $\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}$.
Using the fact that matrix $\rho_1\rho_2$ is similar to matrix $\sqrt{\rho_1}\rho_2\sqrt{\rho_2}$ one can write

$$\sqrt{F(\rho_1,\rho_2)} = \mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} = \sum_{i=1}^N \lambda_i,$$

and since $\mathrm{tr}\rho_1\rho_2 = \mathrm{tr}\sqrt{\rho_1}\rho_2\sqrt{\rho_1} = \sum_{i=1}^N \lambda_i^2$ by squaring the above we get

$$F(\rho_1,\rho_2) = \left(\sum_{i=1}^N \lambda_i\right)^2 = \mathrm{tr}\rho_1\rho_2 + 2\sum_{i<j}\lambda_i\lambda_j.$$

For a give matrix $X \in \mathcal{M}_N$ with eigenvalues $\lambda_1,\dots,\lambda_N$ we define elementary symmetric function $s_m(X)$ as $s_m(\lambda_1,\dots,\lambda_N)$. For example

$$s_1(X) = \mathrm{tr}X,\ s_2(X) = \sum_{i<j}\lambda_i\lambda_j \text{ and } s_3(X) = \sum_{i<j<k}\lambda_i\lambda_j\lambda_k.$$

Using this notion we can write the fidelity as

$$F(\rho_1,\rho_2) = \mathrm{tr}\rho_1\rho_2 + 2s_2(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}).$$

## Bounds for fidelity

In his unpublished work Uhlmann suggested an inequality

$$F(\rho_1,\rho_2) \geq \mathrm{tr}\rho_1\rho_2 + \sqrt{2}\sqrt{(\mathrm{tr}\rho_1\rho_2)^2 - \mathrm{tr}\rho_1\rho_2\rho_1\rho_2}.$$

We define sub-fidelity as

$$E(\rho_1,\rho_2) = \mathrm{tr}\rho_1\rho_2 + \sqrt{2}\sqrt{(\mathrm{tr}\rho_1\rho_2)^2 - \mathrm{tr}\rho_1\rho_2\rho_1\rho_2}.$$

Using elementary symmetric functions this quantity can be represented as

$$E(\rho_1,\rho_2) = \mathrm{tr}\rho_1\rho_2 + 2\sqrt{s_2(\rho_1\rho_2)}.$$

This bound is a consequence of the subadditivity of the square root.

We also introduce upper bound which is complementary to sub-fidelity.

$$F(\rho_1,\rho_2) \leq \mathrm{tr}\rho_1\rho_2 + \sqrt{(1-\mathrm{tr}\rho_1^2)(1-\mathrm{tr}\rho_2^2)}.$$

Again we can use elementary symmetric function to get compact expression for super-fidelity

$$G(\rho_1,\rho_2) = \mathrm{tr}\rho_1\rho_2 + \sqrt{(1-\mathrm{tr}\rho_1^2)(1-\mathrm{tr}\rho_2^2)} = \mathrm{tr}\rho_1\rho_2 + 2\sqrt{s_2(\rho_1)s_2(\rho_2)}$$

and thus we have

$$s_2(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}) \leq \sqrt{s_2(\rho_1)s_2(\rho_2)}.$$

## Properties of sub- and super-fidelity

Sub- and super-fidelity share some properties with fidelity.

**i') Bounds:** $0 \leq E(\rho_1,\rho_2) \leq 1$ and $0 \leq G(\rho_1,\rho_2) \leq 1$.

**ii') Symmetry:** $E(\rho_1,\rho_2) = E(\rho_2,\rho_1)$ and $G(\rho_1,\rho_2) = G(\rho_2,\rho_1)$.

**iii') Unitary invariance:** $E(\rho_1,\rho_2) = E(U\rho_1U^\dagger, U\rho_2U^\dagger)$ and $G(\rho_1,\rho_2) = G(U\rho_1U^\dagger, U\rho_2U^\dagger)$, for any unitary $U$.

**iv') Concavity:** Sub- and super–fidelity are concave,

$$E(A, \alpha B + (1-\alpha)C) \geq \alpha E(A,B) + (1-\alpha)E(A,C),$$
$$G(A, \alpha B + (1-\alpha)C) \geq \alpha G(A,B) + (1-\alpha)G(A,C).$$

**v')** Super-fidelity (just like $\sqrt{F}$) is **jointly concave** in its two arguments *i.e.* for $a \in [0,1]$ we have

$$\sqrt{F}(a\rho_1 + (1-a)\rho_2, a\rho_1' + (1-a)\rho_2') \geq a\sqrt{F}(\rho_1,\rho_1') + (1-a)\sqrt{F}(\rho_2,\rho_2').$$

Both $E$ and $G$ are not multiplicative. On the other hand

**vi')** Super-fidelity is super-multiplicative:

$$G(\rho_1\otimes\rho_2, \rho_3\otimes\rho_4) \geq G(\rho_1,\rho_3)G(\rho_2,\rho_4),$$

**vii')** Sub-fidelity jest sub-multiplicative

$$E(\rho_1\otimes\rho_2, \rho_3\otimes\rho_4) \leq E(\rho_1,\rho_3)E(\rho_2,\rho_4).$$

## Difference $F-G$ and $F-E$

$F$ and $G$ coincide if one of the states is pure, but it is natural to ask how big the difference $G-F$ might be. Let us use the Hilbert space of dimension $N = 2M$ and states $\rho_1 = \frac{2}{N}\mathrm{diag}(1,\dots,1,0,\dots,0)$ and $\rho_2 = \frac{2}{N}\mathrm{diag}(0,\dots,0,1,\dots,1)$.

Since they are supported by orthogonal subspaces their fidelity vanishes, $F(\rho_1,\rho_2) = 0$. On the other hand their super–fidelity is equal to

$$G(\rho_1,\rho_2) = \frac{N-2}{N}$$

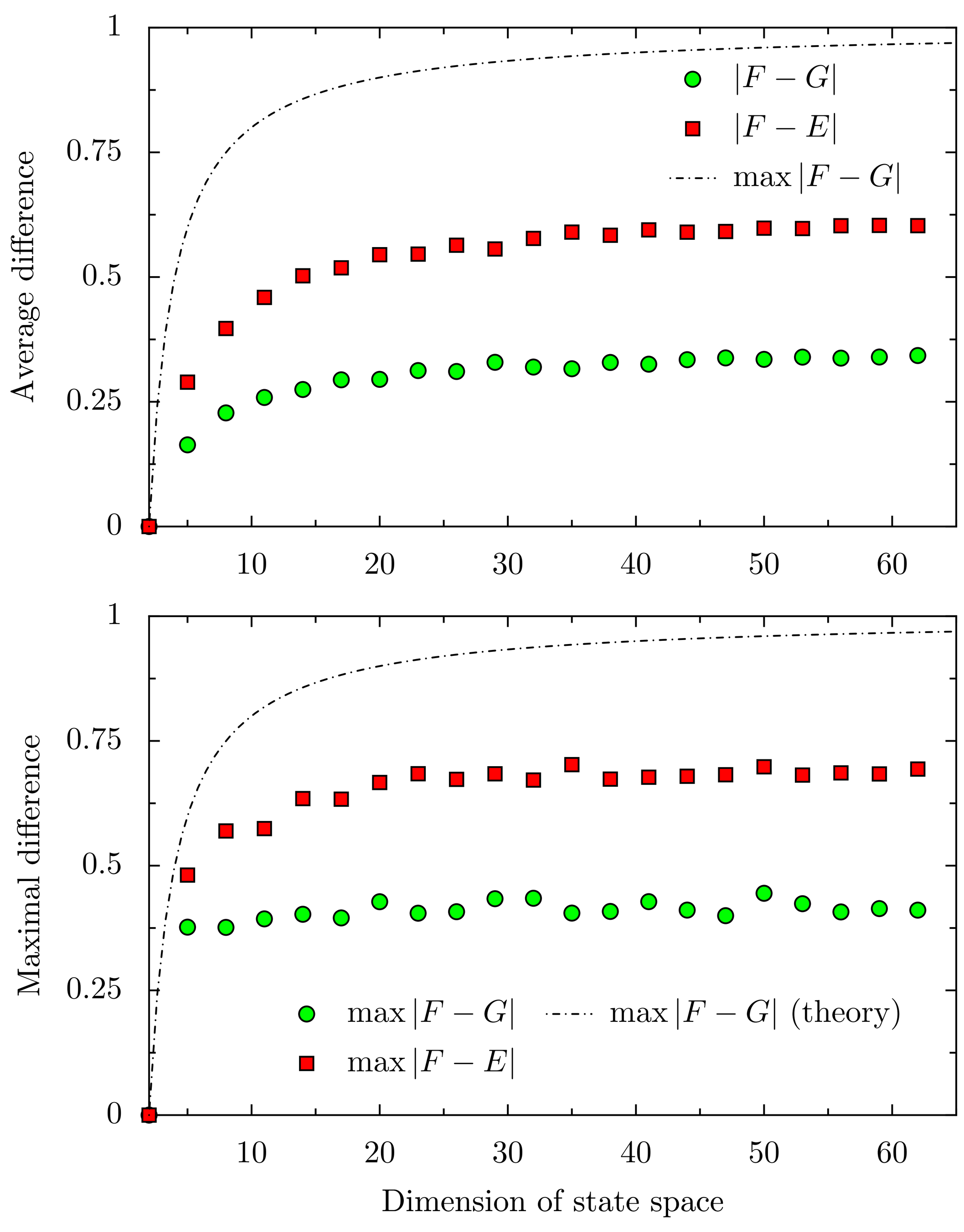


Figure 1: Difference between super-fidelity (sub-fidelity) and fidelity for random states of dimension $N = 2,3,\dots,62$. The upper plot presents average difference and the lower plot show maximal difference.

## Calculation time

Quantities $E$ and $G$ are much easier to calculate than the original fidelity $F$. To compute any of these bounds it is enough to evaluate three traces only. Thus one can expect, that introduced quantities might become useful for various tasks of the theory of quantum information processing.

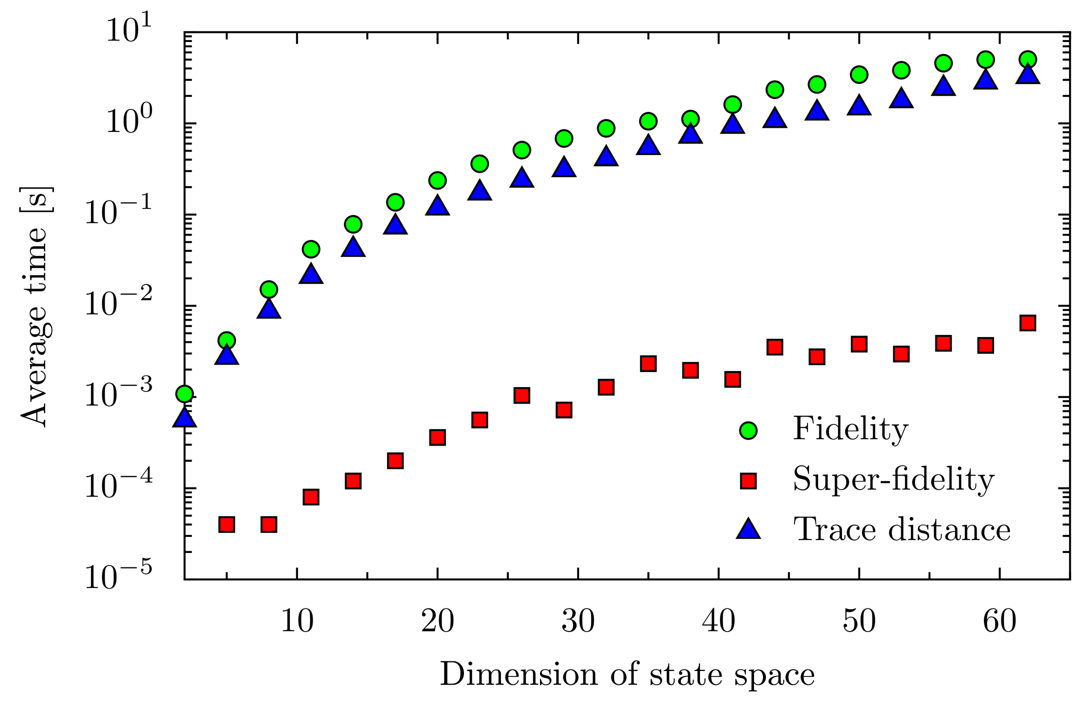


Figure 2: Comparison of calculation time for fidelity $F$, super-fidelity $G$ and trace distance $D_{\mathrm{tr}}$. Data for sample of 100 states for each of dimension $N = 2,3,\dots,62$.

## Measuring super-fidelity

We can use the fact that $\mathrm{tr}V_{12}\rho_1\otimes\rho_2 = \mathrm{tr}\rho_1\rho_2$ where $V_{12}$ is a SWAP operator. $V_{12} = P_{12}^+ + P_{12}^-$ is hermitian and thus represents an observable.

To measure $G$ we need a source which creates pairs $\rho_i\otimes\rho_j,\ i,j = 1,2$.

The probability of measuring $P_{12}^-$ reads $p_{ij}^- = \mathrm{tr}P_{12}^-\rho_i\otimes\rho_j$ and using it we can write

$$G = 1 - 2(p_{12}^- - \sqrt{p_{11}^- p_{22}^-}).$$

Probability of the event that both detectors click is equal to $p_{ij}^-$. On detectors clicks with $p_{ij}^+ = 1 - p_{ij}^-$. Beam-spliter projects on $P^-$ or $P^+$ [4].

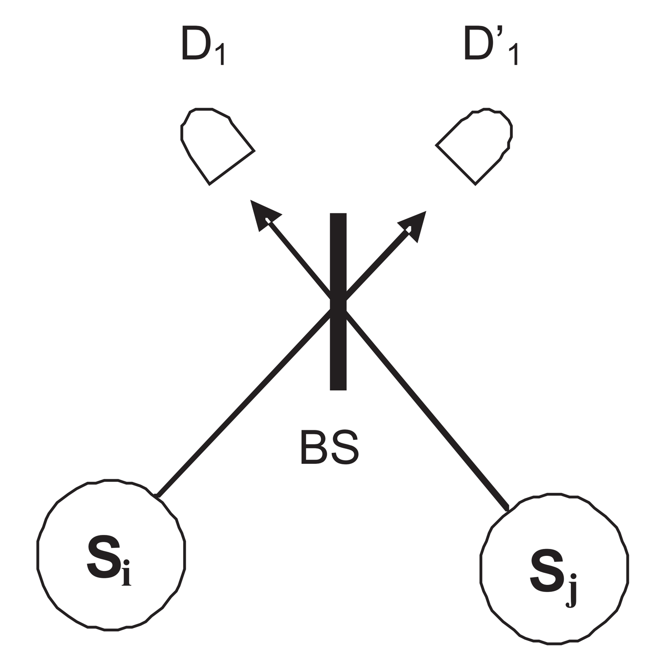


Figure 3: The experimental setup which allows to measure super-fidelity $G$ for qubits. In this case measurement procedure is very simple (see: [1, 4]). This scheme can be extended to the case of $n$ qubits [1].

## References

[1] J. A. M, Z. Puchała, P. Horodecki, A. Uhlmann, K. Życzkowski, *Sub- and super-fidelity as bounds for quantum fidelity*, Quantum Information & Computation, **9**, No.1&2 (2009), arXiv:0805.2037.

[2] P. E. M. F. Mendonca, R. d. J. Napolitano, M. A. Marchiolli, C. J. Foster, and Y.-C. Liang, *An alternative fidelity measure for quantum states*, to appear in PRA, arXiv:0806.1150

[3] J.-L. Chen, L. Fu, A.A. Ungar, and X.-G. Zhao, Alternative fidelity measure between two states of an $N$–state quantum system, *Phys. Rev. A* **65**, 054304 (2002).

[4] F. A. Bovino, G. Castagnoli, A. Ekert, P. Horodecki, C. M. Alves, A. V. Sergienko, Direct Measurement of Nonlinear Properties of Bipartite Quantum States, Phys. Rev. Lett. 95, 240407 (2005)