

NOISE EFFECTS IN QUANTUM MAGIC SQUARES GAME

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In the article we analyse how noisiness of quantum channels can influence the magic squares quantum pseudo-telepathy game. We show that the probability of success can be used to determine characteristics of quantum channels. Therefore the game deserves more careful study aiming at its implementation.

Keywords: Quantum games; quantum channels.

1. Introduction

Quantum game theory, the subclass of game theory that involves quantum phenomena,^{1,2} lies at the crossroads of physics, quantum information processing, computer and natural sciences. Thanks to entanglement, quantum players^a can sometimes accomplish tasks that are impossible for their classical counterparts. In this paper we present the detailed analysis of one of the pseudo-telepathy games.^{3,4} These games provide simple, yet nontrivial, examples of quantum games that can be used to show the effects of quantum non-local correlations. Roughly speaking, a game belongs to the pseudo-telepathy class if it admits no winning strategy for classical players, but it admits a winning strategy provided the players share the sufficient amount of entanglement. This phenomenon is called pseudo-telepathy, because it would appear as magical to a classical player, yet it has quantum theoretical explanation.

^aBy a quantum player we understand a player that, at least in theory, can explore and make profits from the quantum phenomena in situations of conflict, rivalry etc.

Our main goal is to study the connection between errors in quantum channels and the probability of winning in magic squares game. The magic square game is selected because of its unique features: it is easy to show that there is no classical winning strategy, simple enough for a layman to follow its course and feasible.

The paper is organized as follows. We will begin by presenting the magic squares pseudo-telepathy game. Then we will introduce tools we are using to analyse noise in quantum systems. Then we will attempt to answer the following question: *What happens if a quantum game is played in non-perfect conditions because of the influence of quantum noise?* The results showing the connection between the noise level and the probability of winning will be given in Sec. 4. Finally we will point out some issues that yet should be addressed.

2. Magic Square Game

The magic square is a 3×3 matrix filled with numbers 0 or 1 so that the sum of entries in each row is even and the sum of entries in each column is odd. Although such a matrix cannot exist^b one can consider the following game.

There are two players: Alice and Bob. Alice is given the number of the row, Bob is given the number of the column. Alice has to give the entries for a row and Bob has to give entries for a column so that the parity conditions are met. In addition, the intersection of the row and the column must agree. Alice and Bob can prepare a strategy but they are not allowed to communicate during the game.

There exists a (classical) strategy that leads to winning probability of $8/9$. If parties are allowed to share a quantum state they can achieve probability 1.

In the quantum version of this game⁵ Alice and Bob are allowed to share an entangled quantum state.

The winning strategy is following. Alice and Bob share entangled state

$$|\Psi\rangle = \frac{1}{2}(|0011\rangle - |1100\rangle - |0110\rangle + |1001\rangle). \tag{1}$$

Depending on the input (*i.e.* the specific row and column to be filled in) Alice and Bob apply unitary operators $A_i \otimes \mathbb{I}$ and $\mathbb{I} \otimes B_j$, respectively,

$$A_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & i & 1 & 0 \\ 1 & 0 & 0 & i \end{bmatrix} \quad A_2 = \frac{1}{2} \begin{bmatrix} i & 1 & 1 & i \\ -i & 1 & -1 & i \\ i & 1 & -1 & -i \\ -i & 1 & 1 & -i \end{bmatrix} \tag{2}$$

$$A_3 = \frac{1}{2} \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix}$$

^bTherefore the adjective *magic* is used.

$$\begin{aligned}
 B_1 &= \frac{1}{2} \begin{bmatrix} i & -i & 1 & 1 \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & 1 & 1 \end{bmatrix} & B_2 &= \frac{1}{2} \begin{bmatrix} -1 & i & 1 & i \\ 1 & i & 1 & -i \\ 1 & -i & 1 & i \\ -1 & -i & 1 & -i \end{bmatrix} \\
 B_3 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}
 \end{aligned} \tag{3}$$

where i and j denote the corresponding inputs.

The final state is used to determine two bits of each answer. The remaining bits can be found by applying parity conditions.

3. Quantum Noise

A interesting question arises: what happens if a quantum game is played in non-perfect (real-world) conditions because of the presence of quantum noise.^{7,8}

In the most general case quantum evolution is described by superoperator Φ , which can be expressed using Kraus representation⁶:

$$\Phi(\rho) = \sum_k E_k \rho E_k^\dagger, \tag{4}$$

where $\sum_k E_k^\dagger E_k = \mathbb{I}$.

In following we will consider typical quantum channels, namely

- Depolarizing channel: $\left\{ \sqrt{1 - \frac{3\alpha}{4}} \mathbb{I}, \sqrt{\frac{\alpha}{4}} \sigma_x, \sqrt{\frac{\alpha}{4}} \sigma_y, \sqrt{\frac{\alpha}{4}} \sigma_z \right\}$
- Amplitude damping: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\alpha} \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{\alpha} \\ 0 & 0 \end{bmatrix} \right\}$
- Phase flip, bit flip and bit-phase flip with Kraus operators $\left\{ \sqrt{1-\alpha} \mathbb{I}, \sqrt{\alpha} \sigma_z \right\}$, $\left\{ \sqrt{1-\alpha} \mathbb{I}, \sqrt{\alpha} \sigma_x \right\}$ and $\left\{ \sqrt{1-\alpha} \mathbb{I}, \sqrt{\alpha} \sigma_y \right\}$ respectively.

Real parameter $\alpha \in [0, 1]$ represents here the amount of noise in the channel and $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices.

In our scheme, the Kraus operators are of the dimension 2^4 . They are constructed from one-qubit operators e_k by taking their tensor product over all n^4 combinations of $\pi(i)$ indices

$$E_k = \bigotimes_{\pi} e_{\pi(i)}, \tag{5}$$

where n is the number of Kraus operator for a single qubit channel.

3.1. The comparison of channels

Although one can assign physical meaning to the parameter α , this meaning can be different for different channels. Therefore we are using channel fidelity⁹ to compare quantum channels.

We define channel fidelity as:

$$\Delta(\Phi) = F(J(\Phi), J(\mathbb{I})), \tag{6}$$

where J is Jamiołkowski isomorphism and F is the fidelity defined as $F(\rho_1, \rho_2) = \text{tr}(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}})^2$.

4. Results

In this section we are analysing the influence of the noise on success probability and fidelity of non-perfect (mixed) quantum states in the case when the noise operator is applied before the game gates.

4.1. Calculations

The final state of this scheme is $\rho_f = (A_i \otimes B_j) \Phi_\alpha(|\Psi\rangle\langle\Psi|)(A_i^\dagger \otimes B_j^\dagger)$, where Φ_α is the superoperator realizing quantum channel parametrized by real number α .

Depolarizing channel:	
$\{(i, j) i, j = 1, 2, 3\}$	$P_{i,j}(\alpha) = \frac{1}{2}\alpha^4 - 2\alpha^3 + 3\alpha^2 - 2\alpha + 1.$
Amplitude damping channel:	
$(i, j) \in \{(1, 1), (1, 2), (2, 3), (3, 3)\}$	$P_{i,j}(\alpha) = \frac{1}{2}\alpha^2 - \alpha + 1$
$(i, j) \in \{(1, 3)\}$	$P_{i,j}(\alpha) = 2\alpha^2 - 2\alpha + 1$
$(i, j) \in \{(2, 1), (2, 2), (3, 1), (3, 2)\}$	$P_{i,j}(\alpha) = \alpha^2 - \frac{3}{2}\alpha + 1$
Phase damping channel:	
$(i, j) \in \{(1, 1), (1, 2), (2, 3), (3, 3)\}$	$P_{i,j}(\alpha) = \frac{1}{2}\alpha^2 - \alpha + 1$
$(i, j) \in \{(1, 3)\}$	$P_{i,j}(\alpha) = 1$
$(i, j) \in \{(2, 1), (2, 2), (3, 1), (3, 2)\}$	$P_{i,j}(\alpha) = -\frac{1}{2}\alpha + 1$
Phase flip:	
$(i, j) \in \{(1, 1), (1, 2), (2, 3), (3, 3)\}$	$P_{i,j}(\alpha) = 8\alpha^4 - 16\alpha^3 + 12\alpha^2 - 4\alpha + 1$
$(i, j) \in \{(1, 3)\}$	$P_{i,j}(\alpha) = 1$
$(i, j) \in \{(2, 1), (2, 2), (3, 1), (3, 2)\}$	$P_{i,j}(\alpha) = 2\alpha^2 - 2\alpha + 1$
Bit flip:	
$(i, j) \in \{(1, 1), (1, 2), (3, 1), (3, 2)\}$	$P_{i,j}(\alpha) = 2\alpha^2 - 2\alpha + 1$
$(i, j) \in \{(2, 3)\}$	$P_{i,j}(\alpha) = 1$
$(i, j) \in \{(1, 3), (2, 1), (2, 2), (3, 3)\}$	$P_{i,j}(\alpha) = 8\alpha^4 - 16\alpha^3 + 12\alpha^2 - 4\alpha + 1$
Bit-phase flip:	
$(i, j) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$	$P_{i,j}(\alpha) = 2\alpha^2 - 2\alpha + 1$
$(i, j) \in \{(3, 3)\}$	$P_{i,j}(\alpha) = 1$
$(i, j) \in \{(1, 3), (2, 3), (3, 1), (3, 2)\}$	$P_{i,j}(\alpha) = 8\alpha^4 - 16\alpha^3 + 12\alpha^2 - 4\alpha + 1$

Fig. 1. Success probability for all combinations of magic squares game inputs for depolarizing, amplitude damping, phase damping channels, phase, bit and bit-phase flip channels.

Probability $P_{i,j}(\alpha)$ is computed as the probability of measuring ρ_f in the state indicating success

$$P_{i,j}(\alpha) = \text{tr} \left(\rho_f \sum_i |\xi_i\rangle\langle\xi_i| \right), \tag{7}$$

where $|\xi_i\rangle$ are the states that imply success.

4.2. Success probability

We compute success probability $P_{i,j}(\alpha)$ for different inputs $(i, j \in \{1, 2, 3\})$ and different quantum channels. Our calculations show that mean probability of success, $\bar{P}(\alpha) = \sum_{i,j \in \{1,2,3\}} P_{i,j}(\alpha)$, heavily depends on the noise level α . The game results for each combination of gates A_i, B_j for depolarizing, amplitude damping, phase damping, phase, bit and bit-phase flip channels are listed in Fig. 1. Figure 2 presents mean success probability $\bar{P}(\alpha)$ as the function of error rate.

In the case of depolarizing channel, the success probability as the function of noise amount is the same for all the possible inputs. In the case of amplitude and phase damping channels, we can observe three different types of behaviour. These functions are non-increasing for those channels. The bit, phase and bit-phase flip

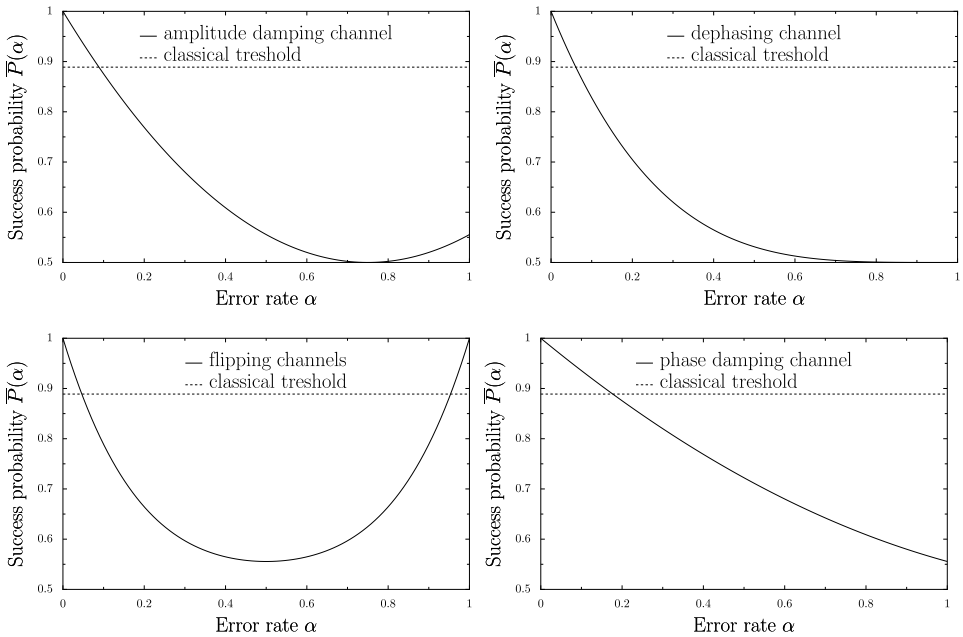


Fig. 2. Dephasing and damping channels cause monotonic decrease of mean success probability in the function of noise amount. Amplitude damping channel causes the success function to attain minimal probability of success for error rate 3/4. Flipping channels give symmetrical functions with minimum for error rate 1/2.

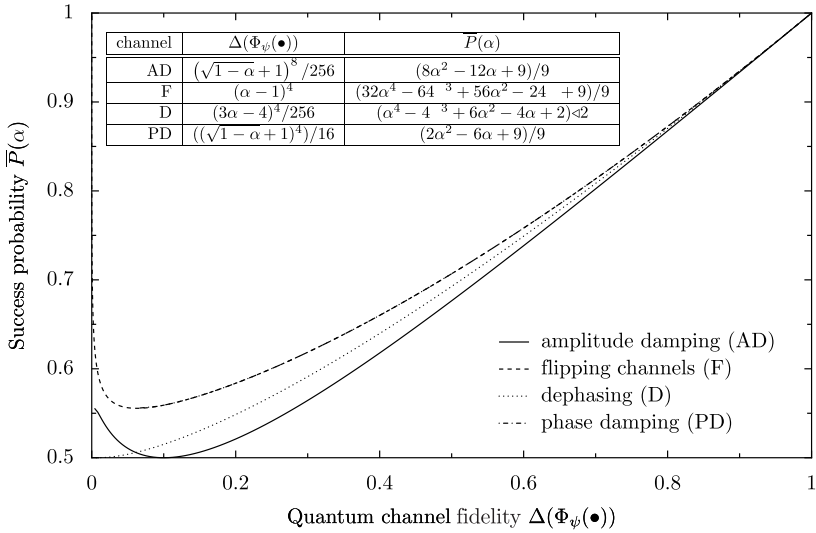


Fig. 3. The influence of noise on success probability, in case of different quantum channels is compared by using the parametric plot of success probability $P_{i,j}(\alpha)$ versus quantum channel fidelity $\Delta(\Phi(\alpha))$ for $\alpha \in [0, 1]$. Note that plots for flipping channels and phase damping channel overall in range $\Delta(\Phi(\alpha)) \in [1/16, 1]$.

functions reach their minima for $\alpha = 1/2$ and are symmetrical. This means that high error rates influence the game weakly. One can easily see that in case of input (1, 3) the phase-flip channel does not influence the probability of success. The same is true for input (2, 3) and bit flip channel and also for input (3, 3) and bit-phase flip channel. Therefore it is possible to distinguish those channels by looking at success probability of magic-squares game.

The graphical representation of dependency between mean success probability and channel fidelity is presented in the form of parametric plot in Fig. 3.

5. Conclusion

We have shown how the probability success in magic squares pseudo-telepathy game is influenced by different quantum noisy channels. The calculations show that, by controlling noise parameter and observing probabilities of success, it is possible to distinguish some channels. Thus we have shown that implementation of magic square game can provide the example of channel distinguishing procedure.

In case of all channels success probability drops, with the increase of noise, below classical limit of 8/9. Therefore the physical implementation of quantum magic squares game requires high precision and can be a very difficult task.

We have also shown that if channel fidelity is higher than 1/10 the probability of success is almost linear. Therefore channel fidelity is good approximation of success probability for not very noisy channels.

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