Quantum states and operations without complex numbers

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Motivation

- Density matrices are hermitian and contain information which is redundant.
- Encoding of data into quantum states requires only real numbers.
- Simplification in *Mathematica* can be very powerful as long as we specify the right assumptions.

Notation

State in quantum mechanics is represented by a positive matrix with trace 1. These properties are reflected by the properties of the quantum maps (*i.e.* linear maps on the space of states).

- $\rho = \rho^\dagger \implies$ Kraus form
- $\rho > 0 \implies$ complete positivity
- $\mathrm{tr}\rho = 1 \implies$ trace preserving

Notation

Quantum operations are linear. As such they can be represented by matrices – so called *supermatrices*. For a quantum state ρ, the action of the map Φ can be expressed as

$$\Phi(\rho) = \mathbf{res}^{-1}\left(M_\Phi\,\mathbf{res}(\rho)\right),$$

where **res** maps $n \times n$ matrices onto n^2-dimensional vectors.

Such representation can be obtained for any linear map on quantum states (not necessary completely positive).

Notation
Real representation

The real representation of a density matrix ρ is defined as

$$\mathcal{R}[\rho]_{ij} = \begin{cases} \mathbf{Re}\rho_{ij} & i \leq j \\ -\mathbf{Im}\rho_{ij} & i > j \end{cases}$$

This translates into *Mathematica* as

```
ComplexToReal[m_] := Block[{d = Dimensions[m][[1]]},
  Array[If[#1 <= #2,
    Re[m[[#1, #2]]],
    –Im[m[[#1, #2]]]] &,
  {d, d}]]
```

Notation

Real representation

One can easily reconstruct the original density matrix from its real representation.

Assuming that m represents a real matrix obtained as a representation of the density matrix, the original density matrix is obtained by

```
RealToComplex[m_] := Block[{d = Dimensions[m][[1]]},
  Array[If[#1 < #2, m[[#1, #2]] + I m[[#2, #1]],
      If[#1 > #2, m[[#2, #1]] - I m[[#1, #2]],
       m[[#1, #2]]]] &, {d, d}]
  ]
```

Notation

Linearity

Maps $\mathcal{R}$ and $\mathcal{C}$ are linear if one considers the multiplication by real numbers only. Thus it can be represented as a matrix on the space of density matrices as

$$\mathcal{R}[\rho] = \mathbf{res}^{-1}\left(M_{\mathcal{R}}\,\mathbf{res}(\rho)\right)$$

where **res** is the operation of reordering elements of the matrix into a vector.

Examples

Example 1: Symbolic density matrices

In *Mathematica* it is easy to express the hermicity of a density matrix

$$\begin{array}{l}\textbf{SymbolicDensityMatrix}[\mathrm{a\_},\ \mathrm{b\_},\ \mathrm{d\_}] := \textbf{Array}[\\ \quad \textbf{If}[\#1 < \#2,\ a_{\#1,\#2} + \textbf{I}\ b_{\#1,\#2},\\ \qquad \textbf{If}[\#1 > \#2,\ a_{\#2,\#1} - \textbf{I}\ b_{\#2,\#1},\ a_{\#1,\#2}]]\ \&,\ \{\mathrm{d},\ \mathrm{d}\}]\end{array}$$

Here a and b are base symbols used to construct symbolic elements of the density matrix.

Examples

Example 1: Symbolic density matrices

However, to use this property during the symbolic manipulations, one has to include the information about $a_{i,j}$ and $b_{i,j}$ in **Assumptions** list

```
$Assumptions = Map[Element[#, Reals] &,
  Flatten[Join[Table[a_{i,j}, {i,1,d}, {j,i,d}],
    Table[b_{i,j}, {i,1,d}, {j, i+1, d}]]]
]
```

This is implemented by **SymbolicDensityMatrixAssume** function, which accepts the same arguments as **SymbolicDensityMatrix**.

Examples

Example 2: One-qubit case

For one qubit the symbolic density matrix can be obtained as **SymbolicDensityMatrix**[a, b, 2] which results in

$$\begin{pmatrix} a_{1,1} & a_{1,2} + ib_{1,2} \\ a_{1,2} - ib_{1,2} & a_{2,2} \end{pmatrix}$$

The list of required assumptions can be obtained as

```
SymbolicDensityMatrixAssume[a, b, 2]
```

Examples

Example 2: One-qubit case

Simplification

In *Mathematica* the application of map $\mathcal{R}$ on the above matrix results in

$$\begin{pmatrix} \mathbf{Re}\left(a_{1,1}\right) & \mathbf{Re}\left(a_{1,2}\right)-\mathbf{Im}\left(b_{1,2}\right) \\ \mathbf{Re}\left(b_{1,2}\right)-\mathbf{Im}\left(a_{1,2}\right) & \mathbf{Re}\left(a_{2,2}\right) \end{pmatrix},$$

and only after using function **FullSimplify** one gets the expected form of the output

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ b_{1,2} & a_{2,2} \end{pmatrix}.$$

Examples

Example 2: One-qubit case

For one-qubit map $\mathcal{R}$ is represented by the matrix

$$M^{(2)}_{\mathcal{R}} = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Matrix representation of the map $\mathcal{C}$ reads

$$M^{(2)}_{\mathcal{C}} = (M^{(2)}_{\mathcal{R}})^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Examples

Example 2: One-qubit case

The main benefit of the real representation is the smaller number of multiplications required to describe the evolution. To illustrate this let us consider a bit-flip channel defined by Kraus operators

$$\left\{ \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix} \right\},$$

or equivalently as a matrix

$$M_{BF}^{(2)} = \begin{pmatrix} 1-p & 0 & 0 & p \\ 0 & 1-p & p & 0 \\ 0 & p & 1-p & 0 \\ p & 0 & 0 & 1-p \end{pmatrix}.$$

Examples

Example 2: One-qubit case

The form of this channel on the real density matrices is given by

$$M_{\mathcal{R}}^{(2)} M_{BF}^{(2)} M_{\mathcal{C}}^{(2)} = \begin{pmatrix} 1-p & 0 & 0 & p \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-2p & 0 \\ p & 0 & 0 & 1-p \end{pmatrix}.$$

This map acts on the real density matrix as

$$\begin{pmatrix} pa_{2,2}-(p-1)a_{1,1} & a_{1,2} \\ (1-2p)b_{1,2} & pa_{1,1}-(p-1)a_{2,2} \end{pmatrix}.$$

Examples

Example 2: One-qubit case

Simplificaiton

In *Mathematica* the direct application of the map $\mathcal{R}$ on the output of the channel, *i.e.* $M_R M_{BF}\,\mathbf{res}\,\rho$, results in

$$\begin{pmatrix} \mathbf{Re}\left(pa_{2,2}-(p-1)a_{1,1}\right) & a_{1,2}+2\mathbf{Im}(p)b_{1,2} \\ (1-2\mathbf{Re}(p))b_{1,2} & \mathbf{Re}\left(pa_{1,1}-(p-1)a_{2,2}\right) \end{pmatrix}$$

To get the simplified result one needs to explicitly specify assumption $\mathrm{p} \in$ **Reals**. This assumption has to be appended to $**Assumptions** variable.

Examples

Example 3: Werner states

Let us consider the Werner states defined for two-qubit systems

$$W(a)=\begin{pmatrix} \frac{a+1}{4} & 0 & 0 & \frac{a}{2} \\ 0 & \frac{1-a}{4} & 0 & 0 \\ 0 & 0 & \frac{1-a}{4} & 0 \\ \frac{a}{2} & 0 & 0 & \frac{a+1}{4} \end{pmatrix},$$

with partial transposition given by

$$W(a)^{T_A}=\begin{pmatrix} \frac{a+1}{4} & 0 & 0 & 0 \\ 0 & \frac{1-a}{4} & \frac{a}{2} & 0 \\ 0 & \frac{a}{2} & \frac{1-a}{4} & 0 \\ 0 & 0 & 0 & \frac{a+1}{4} \end{pmatrix}.$$

Examples
Example 3: Werner states

The real representation reduces one element from the $W(a)$ matrix

$$\mathcal{R}[W(a)] = \begin{pmatrix} \frac{a+1}{4} & 0 & 0 & \frac{a}{2} \\ 0 & \frac{1-a}{4} & 0 & 0 \\ 0 & 0 & \frac{1-a}{4} & 0 \\ 0 & 0 & 0 & \frac{a+1}{4} \end{pmatrix}.$$

This matrix has eigenvalues

$$\left\{ \frac{1-a}{4}, \frac{1-a}{4}, \frac{a+1}{4}, \frac{a+1}{4} \right\}$$

and the sum of smaller eigenvalues is greater than the larger eigenvalue for $a > 1/3$.

Examples

Example 4: Partial transposition

Let us consider the partial transposition of the two-qubit density matrix

$$\rho^{T_A} = \begin{pmatrix} x_{1,1} & x_{1,2}+iy_{1,2} & x_{1,3}-iy_{1,3} & x_{2,3}-iy_{2,3} \\ x_{1,2}-iy_{1,2} & x_{2,2} & x_{1,4}-iy_{1,4} & x_{2,4}-iy_{2,4} \\ x_{1,3}+iy_{1,3} & x_{1,4}+iy_{1,4} & x_{3,3} & x_{3,4}+iy_{3,4} \\ x_{2,3}+iy_{2,3} & x_{2,4}+iy_{2,4} & x_{3,4}-iy_{3,4} & x_{4,4} \end{pmatrix}$$

In this case

$$\mathcal{R}[\rho^{T_A}] = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{2,3} \\ y_{1,2} & x_{2,2} & x_{1,4} & x_{2,4} \\ -y_{1,3} & -y_{1,4} & x_{3,3} & x_{3,4} \\ -y_{2,3} & -y_{2,4} & y_{3,4} & x_{4,4} \end{pmatrix}$$

Examples

Example 4: Partial transposition

However

$$(\mathcal{R}[\rho])^{T_A} = \begin{pmatrix} x_{1,1} & x_{1,2} & y_{1,3} & y_{2,3} \\ y_{1,2} & x_{2,2} & y_{1,4} & y_{2,4} \\ x_{1,3} & x_{1,4} & x_{3,3} & x_{3,4} \\ x_{2,3} & x_{2,4} & y_{3,4} & x_{4,4} \end{pmatrix}$$

so

$$\mathcal{R}[\rho^{T_A}] \neq (\mathcal{R}[\rho])^{T_A}.$$

For this reason one cannot exchange the order of operations.

Examples

Example 4: Partial transposition

The explicit form of the partial transposition on the real density matrices can be found by representing partial transposition as a matrix

```
ChannelToMatrix[PartialTranspose[#, {2, 2}, {1}] &, 4]
```

and using formula

$$M_{\mathcal{R}[\Phi]} = M_{\mathcal{R}} M_{\Phi} M_{\mathcal{C}}.$$

Examples

Example 5: Partial trace

For a two-qubit density matrix its partial trace is given by

$$\mathrm{tr}_A\rho = \begin{pmatrix} x_{1,1}+x_{3,3} & x_{1,2}+x_{3,4}+i\left(y_{1,2}+y_{3,4}\right) \\ x_{1,2}+x_{3,4}-i\left(y_{1,2}+y_{3,4}\right) & x_{2,2}+x_{4,4} \end{pmatrix}.$$

One can verify if the operation of tracing-out the subsystem commutes with the map $\mathcal{R}$ and in this case we have

$$\mathcal{C}[\mathrm{tr}_A\,\mathcal{R}[\rho]] = \mathrm{tr}_A\rho.$$

Thus one can calculate the reduced state of the subsystem using the real value representation.

Summary

- Real quantum states can be used to reduce the amount of memory and the number of operations required during the simulation.
- In some cases (*eg.* partial trace) the calculations can be made using the real representation only.
- Efficient utilization of the simplification procedures requires the appropriate information about the types.
- Order of evaluation and simplification influences the final output.

Thank you!