On the orbit space of unitary actions for mixed quantum states

Vladimir Gerdt, Arsen Khvedelidze, Yuri Palii

Laboratory of Information Technologies,
Joint Institute for Nuclear Research,
141980 Dubna, Moscow Region, Russia
E-mail: gerdt@jinr.ru

Abstract

The space of mixed states, $\mathfrak{P}_+$, of n-dimensional binary quantum system is locus in quo for two unitary groups action: the group $\mathrm{U}(n)$ and the tensor product group $\mathrm{U}(n_1)\otimes\mathrm{U}(n_2)$, where n_1, n_2 stand from dimensions of subsystems, $n = n_1 n_2$. Both groups act on a state $\varrho \in \mathfrak{P}_+$ in adjoint manner $(\mathrm{Ad}\, g)\varrho = g\,\varrho\, g^{-1}$. As a result of this action one can consider two equivalent classes of ϱ; the *global* $\mathrm{U}(n)-$orbit and the *local* $\mathrm{U}(n_1)\otimes\mathrm{U}(n_2)-$orbit. The collection of all $\mathrm{U}(n)$-orbits, together with the quotient topology and differentiable structure defines the "global orbit space", $\mathfrak{P}_+\,|\mathrm{U}(n)$, while the orbit space $\mathfrak{P}_+\,|\mathrm{U}(n_1)\otimes\mathrm{U}(n_2)$ represents the "local orbit space", or the so-called *entanglement space* $\mathcal{E}_{n_1\times n_2}$. The latter space is proscenium for manifestations of the intriguing effects occurring in quantum information processing and communications.

Both orbit spaces admit representations in terms of the elements of integrity basis for the corresponding ring of group-invariant polynomials. This can be done implementing the Processi and Schwarz method, introduced in the 80th of last century for description of the orbit space of a compact Lie group action on a linear space. According to the Processi and Schwarz the orbit space is identified with the semi-algebraic variety, defined by the syzygy ideal for the integrity basis and the semi-positivity condition of a special, so-called "gradient matrix", $\mathrm{Grad}(z) \geqslant 0$, constructing from the integrity basis elements.

In the present talk we address the question of application of this generic computer algebra aided approach to the construction of $\mathfrak{P}_+\,|\mathrm{U}(n)$ and $\mathfrak{P}_+\,|\mathrm{U}(n_1)\otimes\mathrm{U}(n_2)$. Namely, we study whether the semi-positivity of Grad$-$ matrix introduces a new conditions on the elements of the integrity basis for the corresponding ring $\mathbb{R}[\mathfrak{P}_+]^{\mathrm{G}}$.