Controllability of Fractional Linear Systems with Delays in Control

Jerzy Klamka

Abstract. In the chapter linear, fractional, continuous time, finite-dimensional, dynamical control systems with multiple variable point delays and distributed delay in admissible control described by linear ordinary differential state equations are considered. Using notations, theorems and methods taken directly from functional analysis and linear controllability theory, necessary and sufficient conditions for global relative controllability in a given finite time interval are formulated and proved. The main result of the chapter is to show, that global relative controllability of fractional linear systems with different types of delays in admissible control is equivalent to non-singularity of a suitably defined relative controllability matrix. In the proofs of the main results, methods and concepts taken from the theory of linear bounded operators in Hilbert spaces are used. Applying a relative controllability matrix for relative controllable systems steering admissible control is proposed, which steers the fractional system from a given initial complete state to the desired final relative state. Some remarks and comments on the existing controllability results for linear fractional dynamical system with delays are also presented.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory and plays an important role both in traditional and fractional control theory (see e.g. monographs [3], [14], [16], [18], [20]), and survey papers [21], [23], [25]. Controllability is a qualitative property of dynamical control systems and is of particular

---

J.Klamka
Institute of Theoretical and Applied Informatics, Polish Academy of Sciences,
Bałtycka 5, 44-100 Gliwice, Poland
e-mail: jerzy.klamka@iitis.pl
ORCID: 0000-0003-1574-9826

importance in different, mainly theoretical problems in control theory. Systematic study of controllability began at the beginning of the sixties, when the theory of controllability, based on the description in the form of state space for both time-invariant and time-varying linear control systems was presented in [20]. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

In the literature there are many different definitions of controllability, both for linear and nonlinear or semilinear dynamical systems [4], [5], [26], [30], [31], [36]. The concept of controllability strongly depends on the class of dynamical control systems and on the set of admissible controls, [11], [12], [13], [15], [33], [34], [35]. Therefore, for linear and nonlinear or semilinear dynamical systems and fractional systems there exists many different necessary and sufficient conditions for global and local controllability [8], [9], [19], [20]. These conditions are proved using different methods of linear algebra, functional analysis and theory of differential equations and difference equations. Using theory of difference equations and pure algebraic methods, controllability of different discrete time linear fractional control systems is discussed in [7], [13], [14].

The control processes frequently involve different types of delays in state variables or in admissible controls [20]. It should be pointed out, that delay is one of the general phenomenon in a real dynamical system which has a crucial effect on the system properties, for example on the controllability, observability and stability.

Delayed systems constitute a very important class of mathematical models of real phenomena. Delays are inherent in many physical and engineering systems. In particular, pure delays are often used to ideally represent the effects of transmission and transportations. Many applications of delayed systems in engineering, mechanics and economics are presented in the monograph [10]. For dynamical systems with delays in control and/or state variables, two fundamental concepts of states are considered, namely: finite-dimensional instantaneous or relative state and infinite-dimensional complete or functional state [21], [23], [25]. However it should be stressed, that relative state does not provide full information about the trajectory of a control system. Hence it is necessary to introduce at least two different concepts of controllability, namely: relative controllability connected with relative states and complete controllability connected with complete states. Moreover taking into account possible constraints posed on the state variables and on admissible controls [12], local controllability and global controllability are also discussed.

On the other hand, fractional order continuous and discrete mathematical models express the behavior of many real processes more precisely than integer order ones. The various types of fractional differential equations have many applications in different fields of technique for example signal processing, theory of visco-elastic materials [1], [37], supercapacitors [29] filter description and design, circuit theory, computer networks, and bioengineering [15], [16], [17]. Recently different controllability problems have been discussed both for linear and nonlinear fractional infinite dimensional control systems defined in Hilbert spaces. Stochastic boundary controllability of nonlinear fractional systems defined in infinite dimensional Hilbert space is considered in paper [27] using methods of stochastic differential equations. Approximation results for linear fractional diffusion wave equation are presented and discussed in paper [28]. Moreover, the existence and properties of solutions and the initial Cauchy problem for abstract linear differential fractional equations are formulated and discussed in paper [39].

In the present chapter we shall study global relative controllability in a given finite time interval for fractional, linear, continuous time dynamical systems with multiple time variable point delays and distributed delay in admissible control. There are natural generalizations of controllability concepts, which are rather well known in the theory of finite dimensional linear control systems [21], [23], [25] without delays in state variables or in admissible control. Using techniques and methods similar to those presented in monographs [20], [24] and in the series of papers [10], [14], [15] and [19] we shall formulate and prove necessary and sufficient conditions for global relative controllability of linear fractional systems in a prescribed time interval.

This chapter is organized as follows: section 2 contains a mathematical model of a linear, stationary fractional dynamical system with multiple time variable point delays in admissible controls. Moreover, in this section, a basic solution, of a fractional linear finite dimensional differential equation is presented in compact integral form and its properties are also discussed. In section 3 definition of global relative controllability in a given time interval is recalled and discussed. Next, using the results and methods taken directly from linear functional analysis, a global relative controllability problem is mathematically stated and considered. Moreover, using a suitably defined relative controllability matrix, the necessary and sufficient condition and rank condition for global relative controllability in a finite time interval is formulated and proved. The next section 4, is devoted to a study of a popular special case, i.e., relative controllability of fractional systems with multiple constant point delays in admissible control. Necessary and sufficient condition for relative controllability of this system is formulated using results presented in section 3. In section 5, which may be treated as an illustrative example, a linear fractional system with one constant delay in admissible control is considered. In section 6 controllability results for a linear fractional system with distributed delay in admissible control are given. Finally, section 7 contains concluding remarks, and proposes some open controllability problems for more general fractional systems.

2. System Description

Let us consider linear, fractional, delay dynamical systems containing multiple lumped time varying delays in admissible controls, described by the following differential state equation [35], [38], [40]

$$D^{\alpha}x(t) = Ax(t) + \sum_{i=0}^{i=M} B_i(t)u(v_i(t)) \tag{1}$$

for $0 < \alpha \le 1$, and $t \in [t_0 - h, t_1]$.

with initial complete state

$$x(t_0) = x_0 \in R^n, \quad u(t) = u_{t_0}(t), \quad t \in [t_0 - h, t_0] \tag{2}$$

where

$D^{\alpha}(t)$ denotes a fractional Caputo derivative,

$x(t) \in R^n$, is the relative state,

A is $n \times n$ dimensional constant matrix with real coefficients,

B_i, for $i=0,1,\ldots,M$ are given nxp dimensional constant matrices with real coefficients.

admissible controls $u \in U_{ad} = L^2([t_0, t_1], R^p)$.

Initial data $\{x_0, u_{t_0}\}$ forms complete state of the fractional delayed system (1) at initial time t_0.

The strictly increasing and twice continuously differentiable functions $v_i(t):[t_0,t_1] \rightarrow R$, $i=0,1,\ldots,M$, represent deviating arguments in the admissible controls, i.e. $v_i(t)=t-h_i(t)$, where $h_i(t) \ge 0$ are lumped time varying delays for $i=0,1,\ldots M$.

Hence, $v_i(t) \le t$ for $t \in [t_0,t_1]$, and $i=0,1,\ldots,M$, and we assume that $v_0(t)=t$ for $t \in [t_0,t_1]$, and $i=0,1,\ldots,M$.

Let us introduce the time-lead functions $r_i(t):[v_i(t_0),v_i(t_1)]\to[t_0,t_1]$, $i=0,1,\ldots,M$, such that $r_i(v_i(t))=t$ for $t\in[t_0,t_1]$. Furthermore only for simplicity and compactness of notation, let us assume that $v_0(t)=t$ and for a given t_1 the functions $v_i(t)$ satisfy the following inequalities [20].

$$
\begin{aligned}
&h = v_M(t_1) \le v_{M-1}(t_1) \le \ldots v_{m+1}(t_1) \le t_0 = v_m(t_1) < v_{m-1}(t_1) \le \ldots \\
&\ldots \le v_1(t_1) \le v_0(t_1) = t_1
\end{aligned}
\tag{3}
$$

Let us observe, that without loss of generality it may be assumed that $t_0 = v_m(t_1)$.

It is well known (see e.g. [14], [15], [17] or [32]), that for given initial conditions (2) and any admissible control $u\in U_{ad}$, there exists unique solution $x(t;x_0,u)\in L^2([t_0,t_1],R^n)$ of the linear fractional differential state equation (1), which can be represented in the integral form. In order to do that it is convenient to introduce many notations.

The strictly increasing and twice continuously differentiable functions

$$v_i(t):[t_0,t_1]\to R,\ \ i=0,1,\ldots,M$$

represent deviating arguments in the admissible control, i.e., $v_i(t)=t-h_i(t)$, where $h_i(t)>0$ are point, lumped, time varying delays for $i=0,1,\ldots,M$ and

$$v_i(t)\le t,\ t\in[t_0,t_1],\ i=0,1,\ldots,M$$

Let us introduce the time-lead functions

$$r_i(t):[v_i(t_0),v_i(t_1)]\to[t_0,t_1] \text{ for } i=0,1,\ldots,M$$

such that

$$r_i(v_i(t))=t,\ \ t\in[t_0,t_1],\ \ i=0,1,\ldots,M$$

Moreover for a given admissible control function

$$u:[v_M(t_0),t_1]\to R^p$$

where symbol u_t denotes the function defined by the equality

$$u_t(s) = u(t+s) \quad \text{for} \quad s \in [v_M(t), t)$$

For example u_{t_0} denotes the initial admissible control function defined on time interval $[v_M(t_0), t_0)$.

It is well known (see e.g. [14], [15], [19] or [20]), that for given initial conditions (2) and any admissible control $u \in U_{ad}$, there exists unique solution $x(t; x_0, u) \in L^2([t_0, t_1], R^n)$ of the linear fractional differential state equation (1), which can be represented in the integral form.

Furthermore, taking into account linearity of the mathematical model (1) and using Laplace transform method [17], solution $x(t, x_0, u_{t_0}, u)$ of the linear fractional differential equation (1) with the given initial complete state $\{x_0, u_{t_0}\}$ (2) is represented by [14], [16]:

$$\begin{aligned} x(t, x_0, u_{t_0}, u) &= F_0(t-t_0)x_0 + \\ &+ \int_{t_0}^{t} F(t-s) \sum_{i=0}^{i=M} B_i u(v_i(s)) ds, \qquad t \in [t_0, t_1] \end{aligned} \tag{4}$$

where

$$F_0(t) = E_\alpha(At) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}$$

is the Mittag-Leffler $n \times n$ dimensional matrix function for At, [14], [16], [17], where A is $n \times n$ dimensional constant matrix and symbol Γ denotes the Euler gamma function. Similarly we have

$$F(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}$$

Matrix functions $F_0(t)$ and $F(t)$ are used to find the compact integral form of the solution of equation (1) (see e.g. [17], for more details).

Since in this chapter global relative controllability will be considered, let us recall the definition of global relative controllability in a given finite time interval.

Definition 1. The system (1) is said to be globally relatively controllable over time interval $[t_0, t_1]$ if for each pair of vectors $x_0, x_1 \in R^n$ there exists an admissible control $u \in L^2([t_0, t_1], R^m)$ such that the solution of (1) with initial conditions (2) satisfies $x(t_1) = x_1$

Now let us separate from the solution (4) all components which depend on the given initial complete state $\{x(t_0); u_{t_0}\}$. For given final time t_1, using set of inequalities (3) and properties of integrals, it is possible to transform equality (4) as follows:

$$\begin{aligned} x(t_0, t_1, x_0, u_{t_0}, u) &= F_0(t_1 - t_0)x_0 + \\ &+ \sum_{i=0}^{i=m} \int_{v_i(t_0)}^{t_0} F(t_1 - r_i(s))B_i r'_i(s) u_{t_0}(s) ds + \\ &+ \sum_{i=m+1}^{i=M} \int_{v_i(t_0)}^{v_i(t_1)} F(t_1 - r_i(s))B_i r'_i(s) u_{t_0}(s) ds + \\ &+ \sum_{i=0}^{i=m} \int_{t_0}^{v_i(t_1)} \sum_{j=0}^{j=m-i} F(t_1 - r_i(s))B_j r'_j(s)) u(s) ds \end{aligned} \qquad (5)$$

We can divide the right hand side of the formula (5) into two sets. Let us observe that the first three terms on the right-hand side of formula (5) depend only on the initial complete state $\{x_0, u_0\}$ and in fact, do not depend at all on the admissible control $u \in L^2([t_0, t_1], R^m)$. Therefore we can separate these terms and denote shortly as follows:

$$
\begin{aligned}
&q(t_1,t_0,x_0,u_{t_0}) = F_0(t_1-t_0)x_0 + \\
&+\sum_{i=0}^{i=m} \int_{v_i(t_0)}^{t_0} F(t_1-r_i(s))B_i r'_i(s)u_{t_0}(s)ds + \\
&+\sum_{i=m+1}^{i=M} \int_{v_i(t_0)}^{v_i(t_1)} F(t_1-r_i(s))B_i r'_i(s)u_{t_0}(s)ds
\end{aligned} \tag{6}
$$

Thus for given initial data $q(t_1,t_0,x_0,u_{t_0}) \in R^n$ is a constant vector.

Furthermore substituting (6) into (5) we obtain

$$
\begin{aligned}
&x(t_0,t_1,x_0,u_{t_0},u) = q(t_1,t_0,x_0,u_{t_0}) + \\
&+\sum_{i=0}^{i=m} \int_{t_0}^{v_i(t_1)} \sum_{j=0}^{j=m-i} F(t_1-r_j(s))B_j r'_j(s)u(s)ds = \\
&= q(t_1,t_0,x_0,u_{t_0}) + \\
&+\sum_{i=0}^{i=m-1} \int_{v_{i+1}(t_1)}^{v_i(t_1)} \sum_{j=0}^{j=m-i-1} F(t_1-r_j(s))B_j r'_j(s)u(s)ds
\end{aligned}
$$

In order to use results and methods taken directly from the theory of bounded linear operators in Hilbert spaces, let us define linear relative controllability operator [20] as follows:

$$
C_\alpha : L^2([t_0,t_1],R^p) \rightarrow R^n \tag{7}
$$

$$
\begin{aligned}
C_\alpha u &= \sum_{i=0}^{i=m} \int_{t_0}^{v_i(t_1)} \sum_{j=0}^{j=m-i} F(t_1-r_j(s))B_j r'_j(s))u(s)ds = \\
&= \sum_{i=0}^{i=m-1} \int_{v_{i+1}(t_1)}^{v_i(t_1)} \sum_{j=0}^{j=m-i-1} F(t_1-r_j(s))B_j r'_j(s)u(s)ds
\end{aligned} \tag{8}
$$

The range of the relative controllability operator is finite dimensional and since matrix $F(t_1 - r_i(s))$ is bounded for every $t \in [t_0, t_1]$, *i=0,1,...,M,* then C_α is a linear bounded operator.

From relative controllability definition follows, that admissible control *u(t)* steers on the time interval $[t_0, t_1]$, fractional system (1) from the given initial state x_0 to the final state x_1. In fact the relative controllability of (1) is equivalent to finding admissible control $u(t), t \in [t_0, t_1]$, such that for any x_0 and x_1 the following equality holds

$$
\begin{aligned}
& x_1 - q(t_0, t_1 x_0, u_{t_0}) = \\
& = \sum_{i=0}^{i=m} \int_{t_0}^{v_i(t_1)} \sum_{j=0}^{j=m-i} F(t_1 - r_j(s)) B_i r'_i(s) u(s) ds = \\
& = \sum_{i=0}^{i=m-1} \int_{v_{i+1}(t_1)}^{v_i(t_1)} \sum_{j=0}^{j=m-i-1} F(t_1 - r_j(s)) B_j r'_j(s) u(s) ds = C_\alpha(u)
\end{aligned}
\tag{9}
$$

Taking into account the second sum of integrals we see that relative controllability operator C_α is a sum of integral linear bounded operators defined on disjoint sets.

Then the adjoint relative controllability operator C_α^* is also a sum of linear and bounded operators.

In order to find adjoint operator C_α^* let us consider set of integral operators $C_\alpha^i,\ i = 0,1,...,m-1$ given in equality (8). Hence

$$
C_\alpha^i u = \int_{v_{i+1}(t_1)}^{v_i(t_1)} \sum_{j=0}^{j=m-i-1} F(t_1 - r_j(s)) B_j r'_j(s) u(s) ds
$$

for $i = 0,1,...,m-1$.

and the following relation for scalar products in Hilbert spaces:

R^n and $L^2([v_{i+1}(t_1), v_i(t_1)], R^p) = V$

$$
\begin{aligned}
&\left\langle C_\alpha^i u, y \right\rangle_{R^p} = \\
&= \left\langle \int_{v_{i+1}(t_1)}^{v_i(t_1)} \sum_{j=0}^{j=m-i-1} F(t_1 - r_j(s)) B_j r'_j(s) u(s) ds, y \right\rangle_{R^p} = \\
&= \int_{v_{i+1}(t_1)}^{v_i(t_1)} \left\langle u(s), \left(\sum_{j=0}^{j=m-i-1} F(t_1 - r_j(s)) B_j r'_j(s) \right)^* y \right\rangle_V ds = \\
&= \int_{v_{i+1}(t_1)}^{v_i(t_1)} \left\langle \sum_{j=0}^{j=m-i-1} B_j^* F^*(t_1 - r_j(s)) r'_j(s)) u(s), y \right\rangle_V ds = \\
&= \left\langle u(s), C_\alpha^{i*} y \right\rangle_V
\end{aligned}
$$

for $s \in (v_{i+1}(t_1), v_i(t_1)]$ and $i = 0,1,\ldots, m-1$.

Hence every adjoint operator C_α^{i*} for *i=0,1,...,m-1*, which corresponds to C_α^i is defined in different disjoint time intervals as follows

$$
\begin{aligned}
C_\alpha^{i*} y &= \sum_{j=0}^{j=m-i-1} (F(t_1 - r_j(s)) B_j r'_j(s))^* y = \\
&= \sum_{j=0}^{j=m-i-1} B_j^* F^*(t_1 - r_j(s)) r'_j(s)) y
\end{aligned}
\tag{10}
$$

for $s \in (v_{i+1}(t_1), v_i(t_1)]$ and $i = 0,1,\ldots, m-1$.

Therefore, adjoint operator C_α^* is in fact family of operators C_α^{i*} defined on disjoint time intervals $(v_{i+1}(t_1), v_i(t_1)]$ and $i = 0,1,\ldots, m-1$, which covers whole time interval $[t_0, t_1]$.

As was mentioned in the introduction, in this chapter functional analysis methods are used, thus operators C_α and C_α^* defined above, play a crucial role in global relative controllability discussion presented in the next sections.

3. Controllability Conditions

Using the relative controllability operator C_α and its adjoint operator C_α^* let us define the $n\times n$ dimensional relative controllability matrix $W(t_0,t_1)$ for the linear fractional control system

$$W(t_0,t_1)=C_\alpha C_\alpha^* \tag{11}$$

Taking into account relations (8), (9), and (11), relative controllability matrix $W(t_0,t_1)$ is an $n\times n$ dimensional symmetric matrix generally with real coefficients and is defined by the equality:

$$\begin{aligned}
&W(t_1,t_0)=C_\alpha C_\alpha^*=\\
&=\sum_{i=0}^{i=m-1}\int_{v_{i+1}(t_1)}^{v_i(t_1)}\Big(\sum_{j=0}^{j=m-i-1}F(t_1-r_j(s))B_j r'_j(s)\Big)\times\\
&\times\sum_{j=0}^{j=m-i-1}F(t_1-r_j(s))B_j r'_j(s))^* ds=\\
&=\sum_{i=0}^{i=m-1}\int_{v_{i+1}(t_1)}^{v_i(t_1)}\Big(\sum_{j=0}^{j=m-i-1}F(t_j-r_i(s))B_j r'_j(s)\Big)\times\\
&\times\Big(\sum_{j=0}^{j=m-i-1}B_j^* F^*(t_1-r_j(s))r'_j(s)\Big)ds
\end{aligned} \tag{12}$$

Using a relative controllability matrix, it is possible to formulate and prove the main result of the paper given the following theorem, which presents necessary and sufficient conditions for global relative controllability in a given time interval.

Theorem 1. The following statements are equivalent

(1) Fractional system (1) is globally relatively controllable over $t\in[t_0,t_1]$.

(2) Relative controllability operator $C_\alpha : L^2([t_0,t_1],R^p) \to R^n$ is onto.

(3) Adjoint relative controllability operator $C_\alpha^* : R^n \to L^2([t_0,t_1],R^m)$ is invertible i.e., it is one to one operator.

(4) The bounded linear operator $C_\alpha C_\alpha^* : R^n \to R^n$ is onto and may be realized by *nxn* nonsingular matrix.

Proof.

In the proof of Theorem 1, relative controllability linear bounded operator C_α and its adjoint operator C_α^* play an important role. Hence linear functional analysis theory may be applied to prove the theorem. More precisely, we shall use methods and results taken directly from theory of linear bounded operators in Hilbert spaces.

Firstly, let us use, the range of the relative controllability operator C_α that is finite dimensional, and since matrix function *F(t)* is bounded for every $t \in [t_0,t_1]$, then operator C_α is a bounded linear operator. Moreover, as was mentioned before, from the definition 1 and integral formula (8) it immediately follows that the global relative controllability property is equivalent and that relative controllability operator C_α for relatively controllable fractional system (1) is a surjective operator. Hence equivalence (1) and (2) follows.

From the theory of linear operators it follows that surjectivity of the operator C_α implies (see e.g. [9], [11]) that its adjoint operator

$$C_\alpha^* : R^n \to L^2([t_0,t_1],R^m)$$

is also a linear and bounded operator and moreover it is invertible, i.e., “one to one” operator. Hence equivalence (2) and (3) follows

Similarly, from theory of linear bounded operators it follows that invertibility of the selfadjoint operator $C_\alpha C_\alpha^*$ means that there exists inverse operator $(C_\alpha C_\alpha^*)^{-1}$ and is equivalent to surjectivity of the operator C_α. Therefore, for relatively controllable fractional system (1), relative controllability matrix

$$W(t_0,t_1) = C_\alpha C_\alpha^* : R^n \to R^n$$

is invertible, i.e., it is a full rank matrix. Hence equivalence (4) and (1) follows. This statement completes proof of Theorem 1.

From Theorem 1 it follows, that relative controllability matrix $W(t_0,t_1)$ plays a crucial role in relative controllability investigations and moreover, it is also used in admissible control, which transfers initial complete state *x(t₀)* to the final desired relative state x^1 at time t_1.

Let us define admissible control

$$
\begin{aligned}
& u^0(t) = C_\alpha^*(C_\alpha C_\alpha^*)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = \\
& = C_\alpha^* W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = \\
& = \sum_{j=0}^{j=m-i-1} B_j^* F^*(t_1 - r_j(s)) W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0})
\end{aligned}
\tag{13}
$$

for $s \in (v_{i+1}(t_1), v_i(t_1)]$ and $i = 0,1,\ldots,m-1$.

Corollary 1. Admissible control $u^0(t)$ given by formula (13) steers globally relatively controllable fractional system (1) from the given initial complete state $\{x(t_0); u_{t_0}\}$ to the desired final relative state x_1 at time t_1.

Proof.

Substituting equality (13) into solution formula (5) we obtain

$$
\begin{aligned}
& x(t_0,t_1,x_0,u) = q(t_1,t_0,x_0,u_{t_0}) + \\
& + \sum_{i=0}^{i=m} \int_{t_0}^{v_i(t_1)} \sum_{j=0}^{j=m-i} F(t_1 - r_j(s)) B_j r'_j(s) u^0(s) ds = \\
& = C_\alpha C_\alpha^*(C_\alpha C_\alpha^*)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = x_1
\end{aligned}
\tag{14}
$$

Therefore, Corollary 1 is proved.

Remark 1. With the wide class of dynamical systems and especially for dynamical systems with different types of delays, the length of time interval $[t_0,t_1]$ is essential in controllability discussion. It should be pointed out that if the fractional delayed system (1) is globally relatively controllable on the time interval $[t_0, t_1]$, it is also globally relatively controllable on every longer time interval $[t_0, t_2]$, where $t_1 < t_2$ However even for dynamical systems without delays in admissible controls, the opposite statement is not always true, i.e. there are dynamical systems, which are globally controllable on a longer time interval, but are not controllable on a shorter time interval.

Remark 2. Let us observe, that from definition 1 it directly follows that the trajectory of the dynamical system between vectors x_1, and x_2 generally is not prescribed. Therefore, for globally relatively controllable systems generally, there are infinitely many different admissible controls defined on time interval $[t_0, t_2]$, which steer the dynamical system from initial vector x_1, to final vector x_2. However, admissible control $u^0(t)$ defined by formula (13) is optimal in the sense that it has minimum value of energy (see e.g. monographs [2] and [24]), so it is called minimum energy control.

4. Fractional systems with multiple constant delays in control

General results presented in the previous sections may be applied to formulate and to prove necessary and sufficient global relative controllability conditions for systems with multiple constant delays in admissible control. Therefore, now let us consider the special case of a fractional control system (1), i.e. a fractional system with constant multiple delays in admissible control, described by the following equation

$$D^{\alpha} x(t) = Ax(t) + \sum_{i=0}^{i=M} B_i u(t - h_i)\,. \tag{15}$$

$$0 < \alpha \leq 1, \qquad t \in [t_0, t_1]$$

$$0 = h_0 < h_1 < h_2 < \ldots < h_i < h_{i+1} < \ldots < h_{M-1} < h_M$$

In this case

$$v_i(t) = t - h_i \;\; and \;\; r_i(t) = t + h_i\,, \;\; r'_j(t) = 1 \;\; for \;\; i = 0,1,\ldots, M$$

hence,

$$t_1 - h_M < t_1 - h_{M-1} < \ldots < t_1 - h_m = t_0 < \ldots < t_1 - h_1 < t_1 \tag{16}$$

In this case the solution of the linear fractional equation (15) in integral form is given by the following equality

$$\begin{aligned} x(t_0,t_1,x_0,u_{t_0},u) &= F_0(t_1-t_0)x_0 + \\ &+\sum_{i=0}^{i=m}\int_{v_i(t_0)}^{t_0} F(t_1-s-h_i)B_i u_{t_0}(s)ds + \\ &+\sum_{i=m+1}^{i=M}\int_{v_i(t_0)}^{v_i(t_1)} F(t_1-s-h_i)B_i u_{t_0}(s)ds + \\ &+\sum_{i=0}^{i=m}\int_{t_0}^{v_i(t_1)}\sum_{j=0}^{j=m-i} F(t_1-s-h_j)B_j u(s)ds \end{aligned} \tag{17}$$

Moreover operator

$$C_\alpha(u)=\sum_{i=0}^{i=m-1}\int_{v_{i+1}(t_1)}^{v_i(t_1)}\sum_{j=0}^{j=m-i-1} F(t_1-s-h_j))B_j u(s)ds$$

Similarly, as in the previous section, the linear bounded adjoint operator C_α^* is defined as family adjoint operators C_α^{i*} for *i=0,1,...,m-1*, which are defined on a different time interval as follows

$$\begin{aligned} C_\alpha^{i*}y &= \sum_{j=0}^{j=m-i-1}(F(t_1-s-h_j)B_j)^* y = \\ &= \sum_{j=0}^{j=m-i-1} B_j^* F^*(t_1-s-h_j)y \end{aligned} \tag{18}$$

for $s\in(t_1-h_{i+1},t_1-h_i)]$ and $i=0,1,...,m-1$.

Therefore relative controllability matrix $W(t_1,t_0)$ has the form

$$
\begin{aligned}
& W(t_1,t_0) = C_\alpha C_\alpha^* = \\
& = \sum_{i=0}^{i=m-1} \int_{t_1-h_{i+1}}^{t_1-h_i} \Big(\sum_{j=0}^{j=m-i-1} F(t_1-s-h_j)B_j \Big) \times \\
& \times \sum_{j=0}^{j=m-i-1} (F(t_1-s-h_j)B_j)^* ds = \\
& = \sum_{i=0}^{i=m-1} \int_{t_1-h_{i+1}}^{t_1-h_i} \Big(\sum_{j=0}^{j=m-i-1} (F(t_1-s-h_j)B_j) \times \\
& \times \sum_{j=0}^{j=m-i-1} B_j^* F^*(t_1-s-h_j) ds
\end{aligned} \tag{19}
$$

Let us define admissible control

$$
\begin{aligned}
& u^0(t) = C_\alpha^* (C_\alpha C_\alpha^*)^{-1} (x_1 - q(t_0,t_1,x_0,u_{t_0}) = \\
& = C_\alpha^* W(t_0,t_1)^{-1} (x_1 - q(t_0,t_1,x_0,u_{t_0})) = \\
& = \sum_{j=0}^{j=i} B_j^* F^*(t_1-t-h_j) \times \\
& \times W(t_0,t_1)^{-1} (x_1 - q(t_0,t_1,x_0,u_{t_0}))
\end{aligned} \tag{20}
$$

for $s \in (t_1-h_{i+1}, t_1-h_i)]$ and $i = 0,1,...,m.$

Substituting admissible control $u^0(t)$ into solution (17) we obtain

$$
\begin{aligned}
& x(t_0,t_1,x_0,u) = q(t_1,t_0,x_0,u_{t_0}) + \\
& + \sum_{i=0}^{i=m} \int_{t_0}^{v_i(t_1)} \sum_{j=0}^{j=m-i} F(t_1-s-h_j))B_j u^0(s) ds = \\
& = C_\alpha C_\alpha^* (C_\alpha C_\alpha^*)^{-1} (x_1 - q(t_0,t_1,x_0,u_{t_0}) = x_1
\end{aligned}
$$

In the next section, as an illustrative example, relative controllability of a fractional linear control system with only one constant point delay in admissible control will be considered.

5. Fractional system with a single constant point delay in control

In this section let us consider the special case of a fractional control system which is linear. More precisely, we shall discuss the global relative controllability problem for the fractional systems containing only single point constant delay in admissible controls, described by the following fractional differential state equation:

$$D^{\alpha}(t) = Ax(t) + B_0(t)u(t) + B_1(t)u(t-h) \tag{21}$$

$$\text{for } 0 < \alpha \leq 1, \quad t \in [t_0, t_1].$$

with initial complete state

$$x(t_0) = x_0 \in R^n, \quad u(t) = u_{t_0}(t),\ t \in [t_0 - h, t_0] \tag{22}$$

In this case we have

$$v_0(t) = t,\ r_0(t) = t,\ r'_1(t) = 1,\ r_1(t) = t + h, \quad r'_1(t) = 1$$

Since for $t \in [t_0, t_1]$ and $t_1 \leq t_0 + h$ fractional system (21) in fact works as a system without delays in control, let us assume that the final time t_1 satisfies the following inequality, $t_0 + h < t_1$. Similarly, as in the previous sections, in this special case it is also more convincing to present integral relative controllability operator $C_{\alpha}(u)$ in two equivalent simple integral forms:

$$\begin{aligned} C_{\alpha}(u) &= \int_{t_0}^{t_1} F(t_1 - s)B_0 u(s)ds + \int_{t_0+h}^{t_1} F(t_1 - s - h)B_1 u(s)ds = \\ &= \int_{t_0}^{t_0+h} F(t_1 - s)B_0 u(s)ds + \int_{t_0+h}^{t_1} (F(t_1 - s)B_0 + F(t_1 - s - h)B_1)u(s)ds \end{aligned} \tag{23}$$

However from a computations point of view, it is better to take into consideration the second part of the integral formula (23). Thus we obtain adjoint relative controllability operator $C_\alpha^*(u)$ defined by two equalities in two different separated time intervals.

$$
\begin{aligned}
&C_\alpha^* y = ((F(t_1 - t)B_0)^* + (F(t_1 - t - h)B_1)^*)\, y = \\
&= (B_0^* F^*(t_1 - t) + B_1^* F^*(t_1 - t - h))\, y
\end{aligned}
$$

for $t \in (t_0 + h, t_1]$

and

$$C_\alpha^* y = (F(t_1 - t)B_0)^* y = B_0^* F^*(t_1 - t)\, y$$

for $t \in (t_0, t_0 + h]$

Thus the relative controllability matrix is given by the following equality

$$
\begin{aligned}
&W(t_1, t_0) = C_\alpha C_\alpha^* = \\
&= \int_{t_0}^{t_0+h} F(t_1 - s)B_0 B_0^* F^*(t_1 - s)ds + \\
&+ \int_{t_0+h}^{t_1} (F(t_1 - s)B_0 + F(t_1 - s - h)B_1) \times \\
&\times (F(t_1 - s)B_0 + F(t_1 - s - h)B_1)^* ds = \\
&= \int_{t_0}^{t_0+h} F(t_1 - s)B_0 B_0^* F^*(t_1 - s)ds + \\
&+ \int_{t_0+h}^{t_1} (F(t_1 - s)B_0 + F(t_1 - s - h)B_1) \times \\
&\times (B_0^* F^*(t_1 - s) + B_1^* F^*(t_1 - s - h))ds
\end{aligned}
\tag{24}
$$

Using formulas defining operators C_α, C_α^* and matrix $W(t_0,t_1)$ it is possible to find admissible control $u^0(t)$, which steers the initial complete $\{x_0, u_{t_0}\}$ state to the desired final relative state $x_1 \in R^n$ at time t_1.

$$\begin{aligned}
&u^0(t) = C_\alpha^*(C_\alpha C_\alpha^*)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = \\
&= C_\alpha^* W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = \\
&= B_0^* F^*(t_1 - t) W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0})
\end{aligned} \tag{25}$$

for $t \in (t_0, t_0 + h]$

and

$$\begin{aligned}
&u^0(t) = C_\alpha^*(C_\alpha C_\alpha^*)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = \\
&= C_\alpha^* W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = \\
&= (B_0^* F^*(t_1 - t) + B_1^* F^*(t_1 - t - h)) \times \\
&\times W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0})
\end{aligned} \tag{26}$$

for $t \in (t_0 + h, t_1]$

In order to prove the above statement, it is enough to substitute admissible control $u^0(t)$ given by (26) into integral solution (17) of the fractional control system (1). Thus we verify that

$$
\begin{aligned}
&x(t_0,t_1,x_0,u) = q(t_1,t_0,x_0,u_{t_0}) + \\
&+ \int_{t_0}^{t_0+h} F(t_1-s)B_0B_0^*F^*(t_1-s)\times \\
&\times W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0})ds + \\
&+ \int_{t_0+h}^{t_1} (F(t_1-s)B_0 + F(t_1-s-h)B_1)\times \\
&\times (B_0^*F^*(t_1-s) + B_1^*F^*(t_1-s-h))\times \\
&= \times W(t_0,t_1)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0})ds \times \\
&\times C_\alpha C_\alpha^*(C_\alpha C_\alpha^*)^{-1}(x_1 - q(t_0,t_1,x_0,u_{t_0}) = x_1
\end{aligned}
\tag{27}
$$

Hence the desired final relative state is reached.

6. Example

Let us consider the fractional control system (21) with the following data:

$$
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in R^2,\quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ t \in [t_0,t_1] = [0,2] \tag{28}
$$

$$
u(t) \in R,\ u_0(t) = 0,\ t \in [t_0 - h, t_0] = [-1,0]
$$

$$
M = 1,\ B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

For the diagonal matrix A given above, matrix

$$
F(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}
$$

is also a diagonal matrix

$$F(t_1-s)=\begin{bmatrix} f_{11}(t_1-s) & 0 \\ 0 & f_{22}(t_1-s)\end{bmatrix},$$

with positive elements $f_{11}(t_1-s)>0$ and $f_{22}(t_1-s)>0$ on the main diagonal for $s\in[t_0-h,t_1]=[-1,2]$.

Let us consider two cases. The first case is for given final time $t_1\in[t_0,t_0+h]=[0,1]$.

Since both matrices $F(t_1-s)$ and $F^*(t_1-s)$ are diagonal matrices, then in this case relative controllability matrix $W(t_1,t_0)$ given by (24) has the following form

$$\begin{aligned} W(t_1,t_0)&=C_\alpha C_\alpha^*=\int_{t_0}^{t_0+h}F(t_1-s)B_0B_0^*F^*(t_1-s)ds= \\ &=\int_0^1 F(t_1-s)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}F^*(t_1-s)ds=\int_0^1\begin{bmatrix} f_{11}^2(t_1-s) & 0\\ 0 & 0\end{bmatrix}ds \end{aligned} \tag{29}$$

and of course is singular, so taking into account Theorem 1, fractional system (25) is not relatively controllable on the time interval $[0,1]$.

Now let us assume that final time $t_1>t_0+h=1$. In this case relative controllability matrix $W(t_1,t_0)$ is as follows

$$\begin{aligned} &W(t_1,t_0)=C_\alpha C_\alpha^*= \\ &=\int_0^1 F(t_1-s)B_0B_0^*F^*(t_1-s)ds+ \\ &+\int_1^{t_1}(F(t_1-s)B_0+F(t_1-s-h)B_1)\times \\ &\times(B_0^*F^*(t_1-s)+B_1^*F^*(t_1-s-h))ds \end{aligned} \tag{30}$$

Substituting parameters of the fractional system (28) we obtain

$$W(t_0,t_1)=\int_0^1\begin{bmatrix} f_{11}^2(t_1-s) & 0\\ 0 & 0\end{bmatrix}ds+$$

$$+\int_1^{t_1}\begin{bmatrix} f_{11}(t_1-s)\\ f_{22}(t_1-s)\end{bmatrix}\begin{bmatrix} f_{11}(t_1-s) & f_{22}(t_1-s)\end{bmatrix}ds=$$

$$=\int_0^1\begin{bmatrix} f_{11}^2(t_1-s) & 0\\ 0 & 0\end{bmatrix}ds+$$

$$+\int_1^{t_1}\begin{bmatrix} f_{11}(t_1-s)\\ f_{22}(t_1-s)\end{bmatrix}\begin{bmatrix} f_{11}(t_1-s) & f_{22}(t_1-s)\end{bmatrix}ds$$

Therefore, the relative controllability matrix is given by the following formula

$$W(t_0,t_1)=\int_0^1\begin{bmatrix} f_{11}^2(t_1-s) & 0\\ 0 & 0\end{bmatrix}ds+$$

$$+\int_1^{t_1}\begin{bmatrix} f_{11}^2(t_1-s) & f_{11}(t_1-s)f_{22}(t_1-s)\\ f_{11}(t_1-s)f_{22}(t_1-s) & f_{22}^2(t_1-s)\end{bmatrix}ds$$

Hence in this case the relative controllability matrix $W(t_1,t_0)$ is nonsingular and system (28) is globally relatively controllable.

From this example it directly follows that global relative controllability of fractional control systems with delays in admissible controls, strongly depends on the length of the time interval $[t_0,t_1]$.

7. Fractional systems with distributed delays in control

In this section linear, fractional control systems with distributed delays in admissible control are considered. These control systems are extensions of systems with lumped point delays in control and are represented by the following fractional differential state equation [2], [19], [26]

$$D^{\alpha}x(t) = Ax(t) + \int_{-h}^{0} d_{\tau}B(t,\tau)u(t+\tau) \qquad t \in [t_0, t_1] \quad (31)$$

where

$x(t_0) = x_0 \in R^n$ is the given initial condition

$u_{t_0}(t),\ t \in [t_0 - h, t_0]$ is the given initial admissible control

integral term is in the Lebesque-Stieltjes sense [6], [8], [9] with respect to τ,

$B(t,\tau)$ is $n \times p$ dimensional matrix continuous in t for fixed τ and of bounded variation in τ on *[-h,0]* for each $t \in [t_0, t_1]$ and continuous from left in τ on the interval *(-h,0)*.

Using matrices $F_0(t)$ and $F(t)$, which are dependent on α, the solution of differential equation (1) can be expressed in integral form as follows

$$\begin{aligned} x(t_0, t_1, x_0, u_{t_0}, u) &= F_0(t_1)x_0 + \\ &+ \int_{t_0}^{t_1} F(t_1 - s)\left[\int_{-h}^{0} d_{\tau}B(s,\tau)u(s+\tau)\right]ds \end{aligned} \qquad (32)$$

Now, using unsymmetric Fubini theorem (see e.g. [6] and [8] for more details) and changing the order of integration in the last term we have [2], [19], [26]

$$
\begin{aligned}
&x(t_0,t_1,x_0,u_{t_0},u) = F_0(t_1)x_0 + \\
&+\int_{-h}^{0} dB_\tau \left[\int_{t_0}^{t_1} F(t_1-s))B(s,\tau)u(s)ds \right] = \\
&= F_0(t_1)x_0 + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{t_0} F(t_1-(s-\tau))B(s-\tau,\tau)u_{t_0}(s)ds \right] + \\
&+\int_{-h}^{0} dB_\tau \left[\int_{t_0}^{t_1+\tau} F(t_1-(s-\tau))B(s-\tau,\tau)u(s)ds \right] = \\
&= F_0(t_1)x_0 + \int_{-h}^{0} dB_\tau \left[\int_{\tau}^{t_0} F(t_1-(s-\tau))B(s-\tau,\tau)u_{t_0}(s)ds \right] + \\
&+\int_{t_0}^{t_1} \left[\int_{-h}^{0} F(t_1-(s-\tau))d_\tau B_{t_1}(s-\tau,\tau) \right] u(s)ds
\end{aligned}
\tag{33}
$$

where

$$
B_{t_1}(s,\tau) = \begin{cases} B(s,\tau), & s \le t_1, \\ 0, & s > t_1 \end{cases}
$$

The first two terms in formula (33) are dependent on the given initial relative state $\{x_0, u_{t_0}\}$ and in fact do not depend on admissible control $u(t),\ t \ge t_0$. Therefore let us introduce the following notation

$$
\begin{aligned}
&q(t_1,t_0,x_0,u_{t_0}) = F_0(t_1)x_0 + \\
&+\int_{-h}^{0} dB_\tau \left[\int_{t_0+\tau}^{t_0} F(t_1-(s-\tau))B(s-\tau,\tau)u_{t_0}(s)ds \right]
\end{aligned}
\tag{34}
$$

where dB_τ denotes the Lebesque-Stieltjes integration [6], [8], [9] with respect to the variable τ in the matrix function $B(t,\tau)$.

Changing variables in the integral term

$$\left[\int_{-h}^{0} d_\tau B(s,\tau)u(s+\tau)\right]$$

and taking into account the form of solution (34) we obtain

$$\begin{aligned} x(t_0,t_1,x_0,u_{t_0},u) &= q(t_1,t_0,x_0,u_{t_0}) + \\ &+\int_{t_0}^{t_1}\left[\int_{-h}^{0} F(t_1-(s-\tau))d_\tau B_{t_1}(s-\tau,\tau)\right]u(s)ds \end{aligned} \tag{35}$$

Similarly, as in the previous sections, let us introduce relative controllability operator $C_\alpha(t_1)$ and its adjoint operator $C_\alpha^*(t_1)$

$$C_\alpha(t_1)u = \int_{t_0}^{t_1}(\int_{-h}^{0} F(t_1-(s-\tau))d_\tau B_{t_1}(s-\tau,\tau))u(s)ds \tag{36}$$

$$C_\alpha^*(t_1)y = (\int_{-h}^{0} F(t_1-(s-\tau))d_\tau B_{t_1}(s-\tau,\tau))^* y \tag{37}$$

Finally let us define $n\times n$ dimensional relative controllability matrix

$$\begin{aligned} &W(t_0,t_1) = C_\alpha(t_1)C_\alpha^*(t_1) = \\ &= \int_{t_0}^{t_1}(\int_{-h}^{0} F(t_1-(s-\tau))d_\tau B_{t_1}(s-\tau,\tau))(\int_{-h}^{0} F(t_1-(s-\tau))d_\tau B_{t_1}(s-\tau,\tau))^* ds \end{aligned} \tag{38}$$

Corollary 2. Fractional system (31) with distributed delay in admissible control is globally relatively controllable on time interval $[t_0,t_1]$ if and only if the relative controllability matrix (38) is nonsingular.

Proof. From the global relative controllability definition it directly follows, that for relatively controllable fractional system (31) the operator relative controllability operator $C_\alpha(t_1)$ is onto. On the other hand by Theorem 1 this is equivalent, that relative controllability matrix $W(t_0,t_1)$ is nonsingular. Therefore Corollary 2 follows.

For a globally relative controllability fractional system with distributed delay (31) it is possible to find an admissible control, which transforms the given initial complete state to any final relative state at time t_1. Since relative controllability matrix $W(t_0,t_1)$ is a nonsingular matrix then its inverse is well defined. Therefore let us define admissible control as follows

$$
\begin{aligned}
&u^0(t) = C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1 - F_0(t_1)x_0 - \\
&-\int_{-h}^{0} dB_\tau \left[\int_{t_0+\tau}^{t_0} F(t_1-(s-\tau))B(s-\tau,\tau)u_{t_0}(s)ds \right]) = \\
&= C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1 - q(t_0,t_1,x_0,u_{t_0}))
\end{aligned}
\tag{39}
$$

where complete initial state and the final relative state vector are chosen arbitrarily.

Inserting $u^0(t)$ given by (39) into solution formula (33) and taking into account equalities (36), (37) and (38) we have

$$
\begin{aligned}
&x(t_0,t_1,x_0,u_{t_0},u) = q(t_1,t_0,x_0,u_{t_0}) + \\
&+\int_{t_0}^{t_1} \left[\int_{-h}^{0} F(t_1-(s-\tau))d_\tau B_{t_1}(s-\tau,\tau) \right] u^0(s)ds = \\
&= q(t_1,t_0,x_0,u_{t_0}) + \\
&+ C_\alpha(t_1)C_\alpha^*(t_1)W^{-1}(t_0,t_1)(x_1 - q(t_1,t_0,x_0,u_{t_0})) = \\
&= q(t_1,t_0,x_0,u_{t_0}) + \\
&+ W(t_0,t_1)W^{-1}(t_0,t_1)(x_1 - q(t_1,t_0,x_0,u_{t_0})) = x_1
\end{aligned}
\tag{40}
$$

Thus the admissible control $u^0(t)$ transfers the initial complete state to the desired final vector at time t_1.

8. Conclusions

The main result of this chapter is to show and thus prove that global relative controllability of fractional control systems with delays in admissible control is equivalent to non-singularity of a suitably defined square relative controllability matrix.

Using a suitably defined relative controllability matrix for global relatively controllable systems steering admissible control is proposed, which steers the system from the given initial complete state to the desired final relative state. Moreover, at the beginning of the chapter some remarks and comments on the existing literature on controllability results for different types of linear continuous-time and discrete-time fractional dynamical system are also presented.

Using a functional analysis approach, the controllability results presented in this chapter may be extended in many different ways. First of all, using a relative controllability matrix, relative controllability problems for semilinear fractional control systems with different types of delays not only in admissible controls but also in the state variables recently considered in papers [26], [30], [31], [36].

The second possibility is to formulate and prove the necessary and sufficient conditions for relative controllability of fractional control systems with different orders of derivatives, applying methods and concepts proposed in paper [15].

The third direction is to consider infinite dimensional control systems by applying functional analysis methods and concepts (see monographs [20] and [24]). Since in this case, relative state space is infinite dimensional space, then several additional concepts of controllability should be introduced, namely: approximate absolute controllability and exact absolute controllability, approximate relative controllability and exact relative controllability.

In last few years nonlinear or semilinear fractional control systems have been discussed in the literature, e,g. in papers [26], [27], [36]. However so far, only little known reports on global or local relative controllability have been published. It follows from the fact that for nonlinear or semilinear fractional systems, we do not know the exact form of the solution for the nonlinear state equation. Relative controllability conditions for semilinear fractional systems with dominated linear parts are discussed in the paper [36] under the assumption that the linear part is relatively controllable and the nonlinear part satisfies certain inequality.

Generally in the case of semilinear or nonlinear fractional control systems, different techniques are used. The most popular is the fixed-point technique. For example, it is possible to use Banach fixed point theorem, Schauder fixed point theorem, Schaefer fixed point theorem or Darbou fixed point theorem based on measures of noncompactness in Banach, spaces, [12], [14]. It strongly depends on the form of the nonlinear part of the fractional state equation.

Minimum energy control problems similarly as for standard linear systems, are strongly connected with the controllability concept, (see e.g., [16], [20], [24] for more details). First of all, let us observe that for a relatively controllable linear control system there exists generally, many different admissible controls transferring the given initial state complete state to the desired final relative state. Therefore, we may ask which of these possible admissible controls are the optimal one according to given a priori criterion.

For quadratic criterion and relatively controllable linear fractional systems (1), (21) or (31) the solution to this problem can be found using a relative controllability matrix. Moreover, the minimum energy value may be computed in rather simple form. However it should be mentioned, that this method requires many additional restrictive assumptions [20] for example, that state variables and admissible controls are unbounded in the whole time interval.

Acknowledgments. The research was founded by Polish National Research Centre under grant "The use of fractional order controllers in congestion control mechanism of Internet", grant number UMO-2017/27/B/ST6/00145.

References

1. Bagley, R. L., Calico, R. A.: Fractional order state equations for the control of viscoelastically damped structure, Journal of Guidance, Control and Dynamics, 14(2), 304-311, (1991)
2. Balachandran, K., Zhou, Y., Kokila J.: Relative controllability of fractional dynamical systems with distributed delays in control, Computers and Mathematics with Applications, 64(10), 3201-3209, (2012)
3. Balachandran, K., Kokila, J.: On the controllability of fractional dynamical systems, International Journal of Applied Mathematics and Computers Science, (AMCS), 22(3), 523-531, (2012)
4. Balachandran, K., Govindaraj, V., Rdriguez-Germa L., Trujillo J.J.: Controllability results for nonlinear fractional order dynamical systems, Journal of Optimization Theory and Applications (JOTA), 156(1), 33-44, (2013)
5. Balachandran, K., Govindaraj, V., Rdriguez-Germa L., Trujillo J.J.: Controllability of nonlinear higher order fractional dynamical systems, Nonlinear Dynamics, 71(4), 605-612, (2013)
6. Billingsley, P.,: Probability and Measure, chapter 3 Integration, section 18 Product Measure and Fubini's Theorem, Wiley, New York, 231-240, (1995)
7. Busłowicz, M.: Controllability, reachability minimum energy control of fractional discrete-time linear systems with multiple delays in state, Bulletin of the Polish Academy of Sciences. Technical Sciences, 62(2), 233-239, (2014)
8. Cameron, R. H., Martin, W. T.,: An unsymmetric Fubini theorem, Bulletin of the American Mathematical Society, 47(2), 121-125, (1941)

9. Dunford, N., Schwartz T..J.: Linear Operators. Part 1. Chapter 8, Operators and Their Applications. Wiley, New York, (1988)
10. Górecki, H.: Analysis and Synthesis of Time Delay Systems, Wiley, New York, (1989
11. Kaczorek, T.: Rechability and controllability to zero of cone fractional linear systems, Archives of Control Sciences 17(3), 357-367 (2007)
12. Kaczorek, T.: Fractional positive continuous-time linear systems and their reachability”. International Journal of Applied Mathematical Computation Science 18(2), 223-228 (2008)
13. Kaczorek, T.: Positive fractional 2D continuous-discrete time linear systems, Bulletin of the Polish Academy of Sciences, Technical Sciences, 59(4), 575-579 (2011)
14. Kaczorek, T: Selected Problems of Fractional Systems Theory, Springer-Verlag, Berlin, (2012)
15. Kaczorek, T.: Positive linear systems with different fractional orders, Bulletin of the Polish Academy of Sciences, Technical Sciences, 58(3), 453-458, (2010)
16. Kaczorek, T., Rogowski, K.: Fractional Linear Systems and Electrical Circuits, Studies in Systems, Decision and Control, vol. 13, Springer, (2015)
17. Kexue, L., Jigen, P.: Laplace transform and fractional differential equations, Applied Mathematic Letters, 24(12), 2019-2023, (2011)
18. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, (2006)
19. Klamka, J.: Relative controllability and minimum energy control of linear systems with distributed delays in control, IEEE Transactions on Automatic Control, 21(4), 594-595, (1976)
20. Klamka, J.: Controllability of Dynamical Systems, Kluwer Academic, Dordrecht, (1991)
21. Klamka, J.: Controllability of dynamical systems - a survey, Archives of Control Sciences 2(3-4), 281-307 (1993)
22. Klamka, J.: Controllability and minimum energy control problem of fractional discrete-time systems, chapter in monograph “New Trends in Nanotechnology and Fractional Calculus”, editors: D. Baleanu, Z.B. Guvenc and J.A Tenreiro Machado. Springer-Verlag. New York 503-509 (2010)
23. Klamka, J.: Controllability of dynamical systems. A survey, Bulletin of the Polish. Academy of Sciences,. Technical Sciences,. 61(2), 221-229 (2013)
24. Klamka, J.: Controllability and Minimum Energy Control, Studies in Systems, Decision and Control, vol. 162, pp. 1-175, Springer Verlag, (2018)
25. Klamka, J.: Controllability of dynamical systems - a survey, Archives of Control Sciences, 2(3/4), .281-307, (1993)
26. Kumar, P., S., Balachandran, K., Annapoorani, N.: Relative controllability of nonlinear fractional Longevin systems with delays in control, Vietnam Journal of Mathematics, https://doi.org/10.1007/s10013-019-00356-4, (2019)

27. Lizzy, R.M., Balachandran, K.: Boundary controllability of nonlinear stochastic fractional systems in Hilbert spaces, International Journal of Applied Mathematics and Computers Science, (AMCS), 28(1), 123-133, (2018)
28. Mitkowski, W.: Approximation of fractional diffusion wave equation, Acta Mechanica et Automatica, 5(.2), 2011, 65-68, (2011)
29. Mitkowski, W., Skruch, P.: Fractional-order models of the supercapacitors in the form of RC ladder networks, Bulletin Polish Academy of Sciences, Technical Sciences, 61(3),.581-587, (2013)
30. Nirmala, R. J., Balachandran, K., Rdriguez-Germa L., Trujillo J.J.: Controllability of nonlinear fractional delay dynamical systems, Reports on Mathematical Physics, 77(1), 87-104, (2016)
31. Nirmala, R. J., Balachandran, K.: The control of nonlinear implicit fractional delay dynamical systems, International Journal of Applied Mathematics and Computers Science, (AMCS), 27(3), 501-513, (2017)
32. Ostalczyk, P., The non-integer difference of the discrete-time function and its application to the control system synthesis, International Journal of Systems Sciences. 31(12), 1551-1561, (2000)
33. Sikora, B., Klamka J.: Constrained controllability of fractional linear systems with delays in control, Systems and Control Letters, 106(1), 9-15, (2017)
34. Sikora, B., Klamka J.: Cone-type constrained relative controllability of semilinear fractional systems with delays, Kybernetika, 53(2), 370-381, (2017)
35. Sikora, B., Klamka, J.: New controllability criteria for fractional systems with varying delays, in Theory and Applications of Non-integer Order Systems. Editors: Babiarz Artur, Czornik Adam, Klamka Jerzy, Niezabitowski Michał. Lecture Notes in Electrical Engineering, vol. 407, Springer Verlag, (2017)
36. Sivabalan, M., Sathiyanathan, K.: Relative controllability results for nonlinear higher order fractional delay integrodifferential systems with time varying delay in control, Communications Faculty of Sciences, University of Ankara, series A1, Mathematics and Statistics, 68(1), 889-906, (2019)
37. Torvik, P. J., Bagley, R.L.: On the appearance of the fractional derivative in the behavior of real materials, Journal of Applied Mechanics, 51(2), 294-298, (1984)
38. Venkatesan, G., George, R.: Controllability of fractional dynamical systems. A functional analytic approach. Mathematical Control and Related Fields. 7(4), 537-562, (2017)
39. Wang, J. R., Zhou, Y., Feckan, M.: Abstract Cauchy problem for fractional differential equations, Nonlinear Dynamics, 71(4), 685-700, (2013)
40. Wei, J.: The controllability of fractional control systems with control delay, Computers and Mathematics with Applications, 64(10), 3153-3159, (2012)