On Stabilizability of Discrete Time Systems with Delay in Control

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Abstract. The paper deals with discrete time-invariant systems with a delay in the control variable. The relations between different types of controllability and stabilizability are presented and discussed. The results are related to asymptotic null controllability, bounded feedback stabilizability and small feedback stabilizability for linear discrete-time systems with delay in control. The main tool employed is the technique of reducing the delayed equation to a delay-free equation. Thanks to this idea the criteria for bounded feedback stabilizability and small feedback stabilizability for the delayed systems are expressed in the appropriate properties of delay-free systems. Main results are analogical of this one proved in [18] for discrete time-invariant delay-free systems and to those from [14] for continuous-time systems. One of the additional result of this paper provides a criterion for controllability of discrete time system with delay in control. An important contribution of this paper is the indication of further generalizations of the obtained results.

Keywords: Stabilizability · Controllability · Delayed system · Discrete system

1 Introduction

One of the fundamental problems in control theory is designing of an efficient feedback which guarantees control strategy capable to stabilize the possibly unstable system and guarantee a certain level of performance. This problem is well-known and it was extensively investigated in the literature [2,4,13,14] for the delay-free systems.

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However, delay system constitute, nowadays, an important class of mathematical models of real phenomena. Many applications of delayed systems in engineering, mechanics and economics are presented in [5]. Delay is very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. [11]. Delays are inherent in many physical and engineering systems. In particular, pure delays are often used to ideally represent the effects of transmission and transportation. This is because these systems have only limited time to receive information and react accordingly. Such a system cannot be described by purely differential equations, but has to be treated with differential-difference equations or the so-called differential equations with difference variables. The basic theory concerning the stability of systems described by equations of this type was developed by Pontryagin in 1942. Also, important work has been written by Bellman and Cooke in 1963, [3]. The presence of time delays in a feedback control system leads to a closed-loop characteristic equation which involves the exponential type transcendental terms. The exponential transcendentally brings infinitely many isolated roots, and hence it makes the stability analysis of time-delay systems a challenging task. It is well recognized that there is no simple and universally applicable practical algebraic criterion, like the Routh-Hurwitz criterion for stability of delay-free systems, for assessing the stability of linear time-invariant time-delayed systems. On the other side, the existence of pure time delay, regardless if it is present in the control or/and state, may cause an undesirable system transient response, or generally, even an instability. Numerous reports have been published on this matter, with a particular emphasis on the application of Lyapunovs second method, or on using the idea of matrix measure [8]. The analysis of time-delay systems can be classified such that the stability or stabilization criteria involve the delay element or not. In other words, delay independent criteria guarantee global asymptotic stability for any time-delay that may change from zero to infinity.

This arguments motivate us to investigate certain properties of discrete linear equation with delay in control. We study asymptotic null controllability, bounded feedback stabilizability and small feedback stabilizability. The main idea we use is to convert the delay equation into a delay-free equation.

We call a system asymptotically null controllable with bounded control if there exists a control, which steers the solution asymptotically to origin. A natural question is whether the control can be realised in a feedback form or in a form of feedback with small feedback gain. This questions are the main topic of this paper. References closely related to these questions are [10, 16]. In these papers results on semi-global stabilizability and global stabilizability for some special systems are presented for delay-free systems. In the context of stabilization of discrete-time systems is worth to notice the paper [15], where the authors consider the problem of stabilization of a linear time-invariant system given by transfer function, by a first-order feedback controller. The paper contains a description of stabilizing controller in the controller parameter space. The solution is describe by Chebyshev representations of the characteristic equation in the unit circle. Moreover, it is shown that the set can be computed explicitly. Also stabilization of discrete-time invariant system with delay is investigated in [19]. The main results of this paper are about observer based output feedback stabilization. Based on predictor feedback theory a design method of the controller is proposed. Two classes of controllers, namely, the memory observer and memory less observer are considered. The problem of stabilization of time-invariant discrete linear systems with time-varying delay is considered in [17], where the authors use the H_∞ approach to obtain conditions for the output feedback stabilization. The paper [9] focuses on the stabilization problem of discrete linear time-invariant systems with multiply delays in the control variable and multiplicative noise. The authors first transform the original system into a delay-free system and next use the linear quadratic technique. The main result of this paper states that the system can be stabilized in the mean-square sense if and only if the set of solutions of the Riccati difference equations is convergent.

In the present paper we investigate a discrete time-invariant linear system with delay in the control variable and we describe relations between controllability and stabilizability. Following the idea from [9] we use the technique of converting the original delay system to a delay-free one. In the best knowledge of the authors such questions for discrete-time systems with delay have not been investigated in the literature. Similar questions for continuous-time systems with delay have been investigated in [1,7,12].

2 Main Results

In this paper we consider the following equations

$$x(k+1) = Ax(k) + B_0u(k) + B_1u(k-1) \tag{1}$$

$$y(k+1) = \hat{A}y(k) + \hat{B}v(k) \tag{2}$$

$k = 0, 1, ...,$ where the states $x(k)$, $y(k) \in \mathbb{R}^n$, controls $u(k) \in \mathbb{R}^m$, $v(k) \in \mathbb{R}^p$ and A, $\hat{A}$, B_0, B_1, $\hat{B}$ are given matrices of appropriate sizes. In case of Eq. (1) we put $u(-1) = 0$. For initial condition $x(0) = x_0 \in \mathbb{R}^n$, $y(0) = y_0 \in \mathbb{R}^n$ and fixed controls u, v the appropriate solutions of (1) and (2) are denoted by $x(\cdot, x_0, u)$ and $y(\cdot, y_0, v)$. If the control u in (1) has the form $u(k) = L_1(x(k))$ or $u(k) = L_2(x(k), x(k-1))$ for certain functions $L_1 : \mathbb{R}^n \to \mathbb{R}^m$, $L_2 : \mathbb{R}^{2n} \to \mathbb{R}^m$ then we say that u is a feedback control and then we call L_1 and L_2 a feedback. Similarly a control $v(k) = \hat{L_1}(y(k))$ or $\hat{L_2}(y(k), y(k-1))$ in (2), where $\hat{L_1} : \mathbb{R}^n \to \mathbb{R}^p$ and $\hat{L_2} : \mathbb{R}^{2n} \to \mathbb{R}^p$ are called feedback controls for (2) and then $\hat{L_1}$ and $\hat{L_2}$ are called feedbacks. If we apply a feedback $u(k) = L(x(k))$ to system (1) then we obtain system

$$x(k+1) = Ax(k) + B_0L(x(k)) + B_1L(x(k-1)) \tag{3}$$

which is called closed-loop system. In the same way we define the closed-loop system for system (2). If the closed-loop system is stable then the feedback is called a stabilizing feedback.

The main idea of this paper is to connect solutions of (1) and (2) according to the following theorem.

Theorem 1. *Sequence* $x(\cdot,x_0,u)$ *is the solution of* (1) *if and only if* $x(\cdot,x_0,u)$ *is the solution of* (2) *with* $\hat{A}=A$, $\hat{B}=\begin{bmatrix}B_0 \; B_1\end{bmatrix}$, $y(0)=x_0$, $p=2m$ *and*

$$v(k)=\begin{bmatrix}u(k)\\ u(k-1)\end{bmatrix} \quad k=0,1,\dots\,.$$

Proof. Suppose that $x(\cdot,x_0,u)$ is the solution of (1) and let $y(\cdot,y_0,v)$ be the solution of (2) with $\hat{A}=A$, $\hat{B}=\begin{bmatrix}B_0 \; B_1\end{bmatrix}$, $y(0)=x_0, p=2m$ and v is defined as in the theorem. For $k=0$ we have $x(0,x_0,u)=x_0=y(0,y_0,v)$. Suppose that $x(l,x_0,u)=y(l,y_0,v)$ for certain $l\in\mathbb{N}$, then

$$x(l+1,x_0,u)=Ax(l,x_0,u)+B_0u(l)+B_1u(l-1)$$

$$=Ay(l,x_0,v)+\begin{bmatrix}B_0 \; B_1\end{bmatrix}\begin{bmatrix}u(l)\\ u(l-1)\end{bmatrix}$$

$$=\hat{A}y(l,x_0,v)+\hat{B}v(l)=y(l+1,x_0,v).$$

Suppose now that $y(\cdot,y_0,v)$ is the solution of (2) with $\hat{A}=A$, $\hat{B}=\begin{bmatrix}B_0 \; B_1\end{bmatrix}$, $y(0)=x_0, p=2m$ and v is defined as in the theorem. For $k=0$ we have $x(0,x_0,u)=x_0=y(0,y_0,v)$. Suppose that $x(l,x_0,u)=y(l,y_0,v)$ for certain $l\in\mathbb{N}$, then

$$y(l+1,x_0,v)=\hat{A}y(l,y_0,v)+\hat{B}v(k)$$

$$=Ax(l,x_0,u)+\begin{bmatrix}B_0 \; B_1\end{bmatrix}\begin{bmatrix}u(l)\\ u(l-1)\end{bmatrix}$$

$$=Ax(l)+B_0u(l)+B_1u(l-1)=x(l+1,x_0,u).$$

The proof is completed.

Definition 1. *[18] We say that system* (1) *is*

- *asymptotically null controllable with bounded controls (ANCBC) if there is a bounded subset* $\mathcal{U}$ *of* $\mathbb{R}^m$ *which contains zero in its interior such that, for each initial state* $x_0\in\mathbb{R}^n$ *there exists a sequence* $u(\cdot)=u(0),u(1),\dots$ *with all values* $u(t)\in\mathcal{U}$, *which steers the solution* $x(t)$ *asymptotically to the origin i.e.*
$$\lim_{k\to\infty}x(k,x_0,u)=0.$$
- *bounded feedback stabilizable (BFS) if there exists a bounded locally Lipschitz feedback* $L:\mathbb{R}^n\to\mathbb{R}^m$ *that the closed-loop system is asymptotically stable.*
- *small feedback stabilizable (SFS) if for every* $\varepsilon>0$ *there exists a stabilizing feedback* $L:\mathbb{R}^n\to\mathbb{R}^m$ *such that* $\|L(x)\|\le\varepsilon$ *for all* $x\in\mathbb{R}^n$.

In an analogical way we define ANCBC, BFS and SFS for system (2). The next theorem proved in [18] shows that this properties are equivalent for system (2).

Theorem 2. *The following conditions are equivalent*

- (2) *is SFS,*
- (2) *is BFS,*
- (2) *is ANCBC.*

The next three theorems describe relations between ANCBC, BFS and SFS for system (1) and similar properties of system (2).

Theorem 3. *Suppose that system* (1) *is SFS then for* (2) *with* $\hat{A} = A$, $\hat{B} = \left[B_0 \; B_1 \right]$ *for every* $\varepsilon > 0$ *there exists a stabilizing feedback* $\overline{L} : \mathbb{R}^{2n} \to \mathbb{R}^m$, $u(k) = \overline{L}\left(x(k), x(k-1)\right)$ *such that* $\|L(x)\| \leq \varepsilon$ *for all* $x \in \mathbb{R}^{2n}$.

Proof: Let us fix $\varepsilon > 0$. Suppose that system (1) is SFS and let $\overline{L} : \mathbb{R}^n \to \mathbb{R}^m$ be the stabilizing feedback satisfying $\|\overline{L}(x)\| \leq \frac{\varepsilon}{\sqrt{2}}$ for all $x \in \mathbb{R}^n$. Let us define a feedbeck $L : \mathbb{R}^{2n} \to \mathbb{R}^{2m}$in (2) as follows

$$L(x, y) = \left[\overline{L}(x) \; \overline{L}(y) \right], \; x, \; y \in \mathbb{R}^n.$$

For the control $v(k) = L(y(k), y(k-1))$ we know by Theorem 1 that the solutions of (1) and (2) coincide for the same initial condition. Moreover

$$\|L(x, y)\| = \left\| \left[\overline{L}(x) \; \overline{L}(y) \right] \right\| \leq \sqrt{2} \max \left\{ \|\overline{L}(x)\|, \|\overline{L}(y)\| \right\} \leq \varepsilon.$$

The proof is completed.

Theorem 4. *Suppose that system* (1) *is BFS then for* (2) *with* $\hat{A} = A$, $\hat{B} = \left[B_0 \; B_1 \right]$ *there exists a bounded locally Lipschitz feedback* $L : \mathbb{R}^{2n} \to \mathbb{R}^m$ *such that the closed-loop system is asymptotically stable.*

Proof: The proof is analogical as the proof of Theorem 1 additionally we have to use the fact that if $\overline{L} : \mathbb{R}^n \to \mathbb{R}^m$ is bounded and locally Lipschitz, then $L : \mathbb{R}^{2n} \to \mathbb{R}^m$, $L(x, y) = \left[\overline{L}(x) \; \overline{L}(y) \right]$, x, $y \in \mathbb{R}^n$ is also bounded and locally Lipschitz.

Theorem 5. *If system* (1) *is ANCBC then* (2) *with* $\hat{A} = A$, $\hat{B} = \left[B_0 \; B_1 \right]$ *is ANCBC.*

Proof: Suppose that (1) is ANCBC. Let $\mathcal{U}$ be the set from the definition of ANCBC and let us define the subset $\hat{\mathcal{U}} \subset \mathbb{R}^{2m}$ by

$$\hat{\mathcal{U}} = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} : u_1, \, u_2 \in \mathcal{U} \right\}.$$

From the properties of the set $\mathcal{U}$ it is clear that the set $\hat{\mathcal{U}}$ contains zero and it is bounded. We will show that for each $y_0 \in \mathbb{R}^n$ there exist a control sequence $(v(k))_{k \in \mathbb{N}}$, $v(k) \in \hat{\mathcal{U}}$ such that

$$\lim_{k \to \infty} y(k, y_0, v) = 0. \tag{4}$$

Let us fix $y_0 \in \mathbb{R}^n$ and consider a sequence $(u(k))_{k\in\mathbb{N}}$, $u(k) \in \hat{\mathcal{U}}$ for all $k \in \mathbb{N}$, that stabilizes system (1) for initial condition y_0. Let us define a sequence $(v(k))_{k\in\mathbb{N}}$ as follows

$$v(0) = \begin{bmatrix} u(0) \\ 0 \end{bmatrix} \tag{5}$$

and

$$v(k) = \begin{bmatrix} u(k) \\ u(k-1) \end{bmatrix}. \tag{6}$$

From Theorem 1 we know that $x(k, y_0, u) = y(k, y_0, v)$ and therefore (4) holds. The proof is completed.

Finally we will present a result about relations between controllability of systems (1) and (2). We start with the definitions of this concept.

Consider certain subset $\mathcal{G}$ of the set of all sequences which elements are in $\mathbb{R}^m$.

Definition 2. *We say that system* (1) *is $\mathcal{G}$-controllable in time N, $N \in \mathbb{N}$ if for all $x_0, x_1 \in \mathbb{R}^n$ there exists a control $u \in \mathcal{G}$ such that*

$$x(N, x_0, u) = x_1.$$

When $\mathcal{G}$ is the set of all sequences of vectors from $\mathbb{R}^m$, then we will say that (1) *is controllable in time N*

Analogically we define controllability of (2). To formulate the next theorem which presents relation between controllability of systems (1) and (2) let us introduce certain special set $\overline{\mathcal{G}}$ of controls in $\mathbb{R}^{2m}$ consisting of all sequences $(u(k))_{k\in\mathbb{N}}$,

$$u(k) = [u_1(k), ..., u_{2m}(k)]^T$$

satisfying

$$[u_1(k), ..., u_m(k)]^T = [u_{m+1}(k+1), ..., u_{2m}(k+1)]^T.$$

Theorem 6. *System* (1) *is controllable in time N if and only if system* (2) *with $\hat{A} = A$, $\hat{B} = \begin{bmatrix} B_0 & B_1 \end{bmatrix}$ is $\overline{\mathcal{G}}$-controllable in time N.*

Proof. Suppose that system (1) is controllable in time N. Let us fix $x_0, x_1 \in \mathbb{R}^n$ and let $u = (u(k))_{k\in\mathbb{N}}$ be a control such that

$$x(N, x_0, u) = x_1.$$

If we apply control

$$v(k) = \begin{bmatrix} u(k) \\ u(k-1) \end{bmatrix}, \; k = 0, 1, ...$$

in system (2) with initial condition x_0, then according to Theorem 1, solutions of (1) and (2) coincides. In particular

$$x(N, x_0, u) = y(N, x_0, v)$$

and therefore

$$y(N, x_0, v) = x_1.$$

It is also clear that $v \in \overline{\mathcal{G}}$ what implies that (2) with $\hat{A} = A$, $\hat{B} = \begin{bmatrix} B_0 \ B_1 \end{bmatrix}$ is $\overline{\mathcal{G}}$-controllable in time N.

Suppose now that system (2) is $\overline{\mathcal{G}}$-controllable in time N. Let us fix $x_0, x_1 \in \mathbb{R}^n$ and let $v = (v(k))_{k \in \mathbb{N}}$ be a control from $\overline{\mathcal{G}}$ such that

$$y(N, x_0, v) = x_1.$$

Denote

$$v(k) = \begin{bmatrix} u(k) \\ u(k-1) \end{bmatrix}, \ k = 0, 1, ...$$

where $u(k) \in \mathbb{R}^m$, $k = 0, 1, ...$ Notice that this notation is correct since $v \in \overline{\mathcal{G}}$. According to Theorem 1, solutions of (1) and (2) coincides. In particular

$$x(N, x_0, u) = y(N, x_0, v)$$

and therefore

$$x(N, x_0, v) = x_1.$$

The proof is completed.

We will illustrate the last theorem on an example.

Example 1. Consider the following system (1)

$$x(k+1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x(k) + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u(k) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u(k-1). \tag{7}$$

We are going to show that the system is controllable in time 2. According to the Theorem 6 we have to prove that the following delay-free system

$$y(k+1) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} y(k) + \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} v(k) \tag{8}$$

is $\mathcal{G}$-controllable in time 2. Since

$$v(k) = \begin{bmatrix} u(k) \\ u(k) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then $\mathcal{G}$-controllability in time 2 of system (8) is equivalent to controllability in time 2 of the following system

$$z(k+1) = Az(k) + B\nu(k),$$

where:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Finally, controllability in time 2 of the last system follows from the classical Kalman controllability condition (see [6]) since

$$[B \quad AB] = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

3 Conclusions

In this paper we study problems of ANCBC, BFS, SFS and controllability of system (1). The origin of this paper is in the results of [14,18] where the authors obtained a complete picture of the relation between global stabilization and controllability of discrete and continuous delay-free time-invariant systems. Here, we were able only to provide necessary conditions for ANCBC, BFS and SFS of delayed systems in terms of analogical properties of delay-free systems. In that purpose we adapted the idea, from [6], of converting the original systems to an appropriate delay-free system. As it was shown in [14] and [18] for delay-free systems the concept of ANCBC, BFS and SFS are equivalent. The problem of equivalence of ANCBC, BFS and SFS for system (1) is an open problem. Another important direction of further research should be finding of generalizations of our results to time-varying systems as well as to systems with time-varying delay. The goal of this research should be a theory analogical to this which is known for delay-free system (see [14,18]).

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