Repeatability of measurements: Non-Hermitian observables and quantum Coriolis force

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A noncommuting measurement transfers, via the apparatus, information encoded in a system's state to the external "observer." Classical measurements determine properties of physical objects. In the quantum realm, the very same notion restricts the recording process to orthogonal states as only those are distinguishable by measurements. Therefore, even a possibility to describe physical reality by means of non-Hermitian operators should *volens nolens* be excluded as their eigenstates are not orthogonal. Here, we show that non-Hermitian operators with real spectra can be treated within the standard framework of quantum mechanics. Furthermore, we propose a quantum canonical transformation that maps Hermitian systems onto non-Hermitian ones. Similar to classical inertial forces this map is accompanied by an energetic cost, pinning the system on the unitary path.

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I. INTRODUCTION

The no-cloning theorem states that unknown quantum states cannot be copied [1], since no measurement can distinguish arbitrary states with certainty. Similarly, the unitary transfer of information from a quantum system to the measuring device—apparatus—cannot distinguish between nonorthogonal states [2,3]. In contrast, all physical properties of classical systems can be determined with arbitrary precision as the recording process does not perturb the system. To put it differently, classical measurements do not involve any backaction.

Those profound facts rely on a rather natural assumption regarding the physical world—the repeatability of measurements. The latter requires that consecutive measurements should result in the same outcome. Consequently, the demand for all physical observables to be Hermitian seems to be justified from the "first principles" [3]. Therefore, even a possibility to represent observables using non-Hermitian operators should *volens nolens* be excluded as their eigenstates are nonorthogonal [4].

Nevertheless, a variety of experimental findings can be explained by means of non-Hermitian operators. For instance, a spontaneous symmetry breaking observed in Refs. [5,6] has been linked to $\mathcal{PT}$-symmetry, a condition weaker than Hermiticity [7]. Here $\mathcal{P}$ and $\mathcal{T}$ denote the parity and time reflection, respectively, and $\mathcal{PT}$-symmetry guarantees that $[\mathcal{PT}, H] = 0$, where H is the system's Hamiltonian. Additionally, exceptional eigenenergies of complex value have also been measured [8]. Recent years have witnessed great theoretical progress towards the understanding of such non-Hermitian systems [4,9]. It has been shown, for example, that conventional quantum mechanics can be extended to the complex domain [10]. Interesting examples are optical systems with a complex index of refraction [11], tilted optical lattices with defects [12], or systems undergoing topological transitions [13,14]. The latter can serve as realizations of $\mathcal{PT}$-symmetry in Bose-Einstein condensates [15]. Also, many breakthroughs in thermodynamics and statistical physics have been reported for non-Hermitian systems. The Jarzynski equality [16] or the Carnot bound [17] may serve as good examples [7,18].

It is only natural to ask whether such theories are fundamental or provide only an *effective* description of nature [19] (e.g., open systems with balanced loss and gain [5]). In this article we show that the requirement to be able to repeat measurements does not exclude *all* non-Hermitian "observables" from the description of physical reality. We prove that non-Hermitian observables with real spectra are as physical as their Hermitian counterparts. In fact, a formal correspondence between the two classes can be established by means of a quantum canonical transformation [20]. To put it differently, non-Hermitian operators provide a convenient way of representing quantum systems in a physically equivalent way [21,22]. This situation is completely analogous to classical mechanics where classical canonical transformations are used to simplify Hamilton's equations of motion [23].

II. REPEATABILITY OF QUANTUM MEASUREMENTS

Let H be a non-Hermitian observable, i.e., $H^\dagger \neq H$. Without loss of generality we assume that H is the Hamiltonain of a quantum system. To be physically relevant H needs to be *at least* diagonalizable. This requirement assures the existence of an orthonormal [24] basis, $\{|E_n\rangle\}$, and eigenenergies, E_n, that can be measured. However, $|E_n\rangle$ are *not* the eigenvectors corresponding to E_n. These will be constructed shortly. For the sake of simplicity, we further assume that the energy spectrum is discrete and nondegenerate. Therefore, we can write [25]

$$V^{-1} H V = \sum_n E_n |E_n\rangle\langle E_n|, \quad \text{with} \quad E_n \in \mathbb{C}, \tag{1}$$

where $\langle E_n | E_m \rangle = \delta_{nm}$. Generally, H is non-Hermitian, and thus the similarity transformation V is not unitary, i.e., $V^\dagger \neq V^{-1}$. Let us rewrite Eq. (1) as

$$H = \sum_n E_n |\psi_n\rangle\langle\phi_n|, \tag{2}$$

where $|\psi_n\rangle := V|E_n\rangle$ and $\langle\phi_n| := \langle E_n|V^{-1}$. By construction, these states form a *biorthonormal* basis [7,26], that is,

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$\langle\phi_n|\psi_m\rangle = \delta_{nm}$ and $\mathbb{I} = \sum_n |\psi_n\rangle\langle\phi_n|$. Moreover, the eigenvalue problem for H can be stated as

$$H|\psi_n\rangle = E_n|\psi_n\rangle \quad \text{and} \quad \langle\phi_n|H = E_n\langle\phi_n|. \tag{3}$$

We emphasize the importance of this biorthonormal set of vectors used for non-Hermitian quantum systems. It plays a role, both conceptually and computationally, analogous to a complete set of eigenvectors of a Hermitian operator. As a result, the left $\langle\phi_n|$ corresponds to the right $|\psi_n\rangle$ in Dirac notation [27]. Hence, the way probability is assigned to a physical process has to be revisited. For instance, consider a system that is prepared in one of its energy eigenstates, say $|\psi_n\rangle$, and then immediately perturbed by a map, U (e.g., time evolution or a measurement). Then the probability to measure the energy E_m reads

$$p_{nm} = p(|\psi_n\rangle \to U \to |\psi_m\rangle) = |\langle\phi_m|U|\psi_n\rangle|^2. \tag{4}$$

This formula provides a *natural* generalization of the "standard" recipe, $|\langle\psi_m|U|\psi_n\rangle|^2$, for calculating probabilities in Hermitian quantum mechanics [27].

Thus far we have seen that the proper identification of probabilities of the measurement outcomes allows us to include non-Hermitian operators into the usual framework of quantum mechanics rather naturally. This is done, in a physically consistent way, by properly accounting for the fact that non-Hermitian observables have different left and right eigenstates. The probability $p_n = |\langle\phi_n|\psi\rangle|^2$ to find the system in its eigenstate $|\psi_n\rangle$ can also be rewritten, using this state *explicitly*, as

$$p_n = |(\langle\psi_n|g)|\psi\rangle|^2, \quad g = \sum_m |\phi_m\rangle\langle\phi_m|. \tag{5}$$

Thus, the Dirac correspondence between bra and ket vectors can now be understood as $\langle\psi|g \leftrightarrow |\psi\rangle$. Above, g is a positive-definite, invertible linear operator—that is, a metric [28]. Indeed, we have

$$\langle\psi|g|\psi\rangle = \sum_m |\langle\phi_m|\psi\rangle|^2, \quad g^{-1} = \sum_n |\psi_n\rangle\langle\psi_n|, \tag{6}$$

where the first equality expresses positivity and the second expression provides an explicit formula for the inverse map in terms of eigenstates $|\psi_n\rangle$. As a result, assigning probabilities when measuring non-Hermitian "observables" defines a new inner product, namely, $(\psi, g\phi)$. Moreover, a simple calculation shows that $H^\dagger g = gH$ and therefore $(H\psi, g\phi) = (\psi, gH\phi)$ for all states $|\psi\rangle$ and $|\phi\rangle$. This fact can also be interpreted as H is Hermitian with respect to this new inner product.

It is important to realize that non-Hermitian operators such as those in Eq. (1) *can* generate unitary dynamics. This is possible *if and only if* the energy spectrum is real. Therefore, *a priori*, non-Hermitian operators do not necessarily violate the conservation of probability. Note not only that every superposition of states $|\psi_n\rangle$ is allowed but also an arbitrary state $|\psi_0\rangle$ can be expressed in such a manner, $|\psi_0\rangle = \sum_n c_n|\psi_n\rangle$, where $\sum_n |c_n|^2 = 1$ and $|c_n|^2$ is the probability for the system to be found in its eigenstate, $|\psi_n\rangle$. The corresponding $\langle\phi_0|$ is given by

$$\langle\phi_0| = \sum_n c_n^*\langle\phi_n|. \tag{7}$$

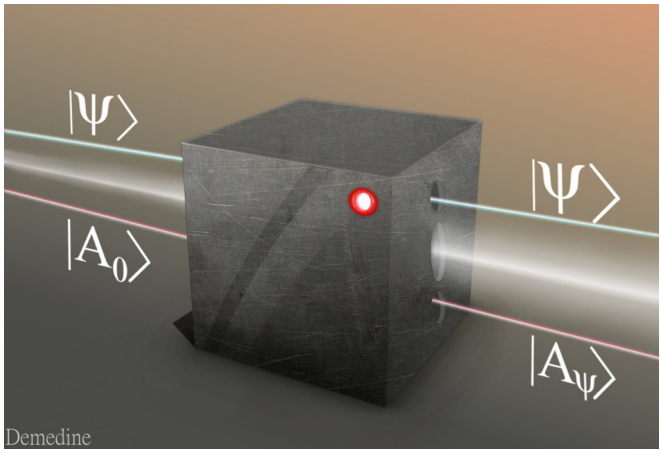

FIG. 1. Schematic representation of the unitary transfer of information from a quantum system $\mathcal{S}$ to the measuring device—apparatus $\mathcal{A}$. Information encoded in the quantum state $|\psi\rangle$ is being recorded by $\mathcal{A}$ and is written down into its state $|A_\psi\rangle$. The recording process $|A_0\rangle \to |A_\psi\rangle$ is unitary and does not influence the information carried by $|\psi\rangle$.

The initial state $|\psi_0\rangle$ evolves under the Schrödinger equation, $i\hbar\partial_t|\psi_t\rangle = H|\psi_t\rangle$. Therefore, the solution reads $|\psi_t\rangle = \sum_n e^{-iE_nt/\hbar}c_n|\psi_n\rangle$, and, as a consequence of Eq. (7), the corresponding $\langle\phi_t|$ evolves according to

$$\langle\phi_t| = \sum_n e^{+iE_n^*t/\hbar}c_n^*\langle\phi_n|. \tag{8}$$

Thus, *if and only if* all eigenvalues E_n are real, then the system's dynamics is unitary and therefore

$$\langle\phi_t|\psi_t\rangle = \sum_n |c_n|^2 e^{i(E_n - E_n^*)t/\hbar} = \langle\phi_0|\psi_0\rangle, \tag{9}$$

proving that the probability is indeed conserved.

We have demonstrated that a fully consistent quantum theory can be built with non-Hermitian operators. We have imposed that "observables" are diagonalizable, which assures that the spectrum can be measured. Its reality, on the other hand, yields unitary dynamics and thus the conservation of probability.

Let $|A_0\rangle$ be a "ready to measure" initial state of the apparatus $\mathcal{A}$. Further, by $|\psi_n\rangle$ and $|\psi_m\rangle$ we denote distinct $(n \neq m)$ eigenstates of H. Also, we assume that $E_n^* = E_n$ and, without loss of generality, we choose $\mathcal{A}$ to be a Hermitian system. The repeatability of measurements guarantees that every unitary transfer of information from system $\mathcal{S}$ to $\mathcal{A}$ leaves states $|\psi_n\rangle$ and $|\psi_m\rangle$ undisturbed. It follows that

$$\mathcal{U} : |\psi_k\rangle|A_0\rangle \to |\psi_k\rangle|A_k\rangle \quad \text{for} \quad k = n, m, \tag{10}$$

where $\mathcal{U}$ is a unitary map (e.g., $\mathcal{U}\mathcal{U}^\dagger = \mathbb{I}$) modeling the recording process. As illustrated in Fig. 1, after the transfer has been completed, new states $|A_m\rangle$ and $|A_n\rangle$ of the apparatus $\mathcal{A}$ encode the information about the system's eigenstates $|\psi_n\rangle$ and $|\psi_m\rangle$. The measurement preserves the norm on the Hilbert space $\mathcal{S} \otimes \mathcal{A}$ as well. Hence

$$\langle\phi_m|\psi_n\rangle(1 - \langle A_m|A_n\rangle) = 0. \tag{11}$$

As a result, the apparatus states $|A_m\rangle$ and $|A_n\rangle$ can differ (indicating some information being stored) only when $\langle\phi_m|\psi_n\rangle = 0$. To put it differently, extracting information that distinguishes between two measured states is possible only when the transition probability between states $|\psi_n\rangle$ and $|\psi_m\rangle$ vanishes.

The above analysis demonstrates that the reality of the spectrum, which guarantees unitarity, rather than Hermiticity is *necessary* to acquire information. We stress that $|\langle\psi_m|\psi_n\rangle|^2$ is *not* the transition probability between states $|\psi_n\rangle$ and $|\psi_m\rangle$. That is given by $|\langle\phi_m|\psi_n\rangle|^2$ and the two coincide only when H is Hermitian as only then the left and right eigenstates are the same.

Adopting the similar formula that was derived for Hermitian systems [2] one could write $\langle\psi_m|\psi_n\rangle(1 - \langle A_m|A_n\rangle) = 0$. As a result, one would have to conclude that $\langle A_m|A_n\rangle = 1$, showing the apparatus cannot tell the measured states apart. However, as we have shown, since H is non-Hermitian, Eq. (11) applies, which allows nonorthogonal states to be measured.

III. RELATION TO HERMITIAN SYSTEMS

As we have seen, from the viewpoint of a measurement, there is no *physical* difference between non-Hermitian operators with real spectra and Hermitian observables. Therefore, one should be able to represent quantum systems either way depending on the situation. Of course, in complete analogy to classical physics the goal is to find the simplest possible Hamiltonian. One can establish a correspondence between H and its Hermitian counterpart K in the following way [29]:

$$\begin{aligned} K &= g^{1/2} H g^{-1/2} = e^{G/2} H e^{-G/2} \\ &= H + \frac{1}{2}[G,H] + \frac{1}{2!2^2}[G,[G,H]] + \cdots \end{aligned} \tag{12}$$

where $G := \ln(g)$. In the second line we have used the Baker-Campbell-Hausdorf-like formula [30]. Note that since the metric g is positive definite its logarithm and square root exist and, moreover, both of these quantities are Hermitian operators [31]. Although Eq. (12) was introduced in Ref. [29] as a similarity map between Hilbert spaces, its physical significance was missed. Generally this infinite series does not truncate after a finite number (see Sec. V B below). We stress that Eq. (12) can be used to transform an arbitrary observable O between Hermitian and non-Hermitian representations.

The first of the above Eqs. (12) shows that K is indeed Hermitian, whereas the second one demonstrates an interesting feature of physical reality. Namely, a quantum system can be represented *equivalently* either by a Hermitian operator or a non-Hermitian one with a real spectrum. Although there is no essential physical difference between the two representations, their mathematical structures are quite different. It follows directly from Eq. (12) that a complicated Hermitian system may have a very simple non-Hermitian representation and *vice versa*. Transformation (12) plays a role analogous to the canonical transformation well established in classical mechanics [23]. Note that this transformation cannot be unitary as it changes the Hermiticity of an operator. However, it preserves the canonical commutation relation and as such belongs to a class of quantum canonical transformations [20,32].

IV. QUANTUM CORIOLIS FORCE

In classical mechanics, Newtonian equations of motion have to be modified in nonstandard, time-dependent, frames of reference [23]. As a result, one observes so-called inertial forces. Typical examples include Coriolis or centrifugal forces that are present *only* in rotating frames of reference. Therefore, there are experimentally accessible consequences of using such noninertial coordinates. One of the most famous examples is the Foucault pendulum whose motion (precession) directly reflects on the Earth's rotation around its own axis [33].

Interestingly, if the non-Hermitian Hamiltonian is time dependent, then the corresponding Schrödinger equation also has to be modified to preserve unitarity [34–36]:

$$i\hbar\partial_t|\psi_t\rangle = (H_t + F_t)|\psi_t\rangle, \quad F_t = -\frac{i\hbar}{2}\Lambda_t^{-1}\partial_t\Lambda_t. \tag{13}$$

Above, Λ_t is a time-dependent metric that does not necessarily coincide with g_t [35]. More importantly, this metric is not unique. However, it can be chosen so that the corresponding Hermitian Hamiltonian K_t in Eq. (12) (i) is the generator of dynamics and (ii) has exactly the same spectrum as H_t.

Therefore, the dynamics in these two representations differ considerably. Nevertheless, if we replace ∂_t with the covariant derivative $D_t := \partial_t + \Lambda_t^{-1}\partial_t\Lambda_t/2$ [37], the Schrödinger equation (13) can be put into its standard form, i.e., with H_t being the generator, $i\hbar D_t|\psi_t\rangle = H_t|\psi_t\rangle$. However, one can also think of the extra energetic contribution $\sim\Lambda_t^{-1}\partial_t\Lambda_t/2$ as being a manifestation of a force of inertia keeping a quantum system along the unitary path during its evolution (see Sec. V C below). We can interpret this force as a quantum Coriolis force.

It is worth mentioning that the existence of F_t in non-Hermitian representations has been noticed yet disregarded as unphysical (see, for example, Ref. [21] and comments that followed). It was treated rather as a mathematical necessity, not having much to do with physical reality. We now illustrate the novel concepts with several analytically tractable examples.

V. EXAMPLES

A. $\mathcal{PT}$ symmetric system

As a first example consider a harmonic oscillator with a non-Hermitian perturbation, for instance [38],

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + i\epsilon x^3 \equiv H_0 + \epsilon H_1, \tag{14}$$

where H_0 corresponds to the unperturbed harmonic oscillator and H_1 is an anharmonic perturbation. Parameters m and ω correspond to the system's mass and the size of the harmonic trap, respectively. Here ϵ is a small perturbation. The momentum p and position x operators obey the standard canonical commutation relation, $[x,p] = i\hbar$. This model has been extensively investigated in literature [39]. Numerical studies have confirmed the reality of its spectrum for all real ϵ. Using perturbation theory one can establish that

$K = H_0 + V(x,p)$, where the momentum-dependent potential $V(x,p)$ up to $O(\epsilon^3)$ reads [29]

$$V(x,p) = \frac{1}{m\omega^4}\left(\{x^2,p^2\} + p\,x^2\,p + \frac{3m\omega^2}{2}\,x^4\right)\epsilon^2, \quad (15)$$

where $\{A,B\} := AB + BA$ is the anticommutator between A and B. Observe that this very complicated, momentum-dependent potential can *effectively* be replaced by a simple non-Hermitian term, $V \sim ix^3$, which only depends on the position x [40].

B. Localization in condensed-matter physics

Another interesting example is a general one-dimensional quantum system whose Hamiltonian reads $K = p^2/2m + V(x)$, where $V(x)$ is an arbitrary potential. This standard textbook Hermitian model can be turned into a very powerful non-Hermitian system that can explain localization effects in solid-state physics [41,42]. Indeed, we have

$$H = e^{-\eta x} K e^{\eta x} = \frac{(p - i\hbar\eta)^2}{2m} + V(x), \quad (16)$$

where the real parameter η expresses an external magnetic field [43]. Note that the metric $g = e^{2\eta x}$ in this case can be calculated explicitly. Furthermore, it depends on the external control parameter—the magnetic field [44]. Also, since the commutator $[x,p] = i\hbar$ is a complex number, the infinite series in Eq. (12) truncates after only two terms.

C. Time-dependent metric and force of inertia

Finally, assume that the metric g from the previous example depends explicitly on time (e.g., the magnetic field η_t is time dependent). We further choose $V(x)$ to be a harmonic trap, $V(x) = m\omega^2 x^2/2$, where ω is its frequency. Then the Hamiltonian in Eq. (16), now time dependent, can be written using the second quantization [45] as

$$H_t = \hbar\omega\left[(a - \eta_t\alpha)^\dagger(a + \eta_t\alpha) + \frac{1}{2}\right], \quad (17)$$

where $\alpha = \sqrt{\hbar/2m\omega}$ and a and $a^\dagger$ are annihilation and creation operators, respectively. As explained above, to preserve unitarity the evolution generator H_t in the non-Hermitian representation has to be modified accordingly. By setting $\Lambda_t = g_t$ [17] in Eq. (13) we have

$$H_t \to H_t + F_t, \quad \text{where} \quad F_t = -i\dot{\eta}_t\alpha(a + a^\dagger). \quad (18)$$

To analyze the evolution of this system we turn to numerical simulations. We further assume that η_t changes on the time scale τ linearly, that is, $\eta_t = t/\tau$ for $0 \leqslant t \leqslant \tau$. The initial state is given by $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, where $|0\rangle$ is the ground state of $K = H_0$ and $|1\rangle = a^\dagger|0\rangle$.

Figure 2 shows the average position $\langle x\rangle$ of a quantum particle as a function of time t computed in both the Hermitian (blue solid line) and the non-Hermitian (red points) representation. According to the Ehrenfest theorem, $\langle x\rangle$ corresponds to the classical trajectory in the sense that it obeys Newton's equations of motion [46]. As we can see, only when the proper energetic contribution F_t is accounted for the two paths coincide. The dashed green line, on the other hand, depicts a nonunitary path resulting from not taking into account this contribution. As expected, F_t does not have any influence on the system's dynamics in nonaccelerating frames of reference where $\dot{\eta}_t = 0$.

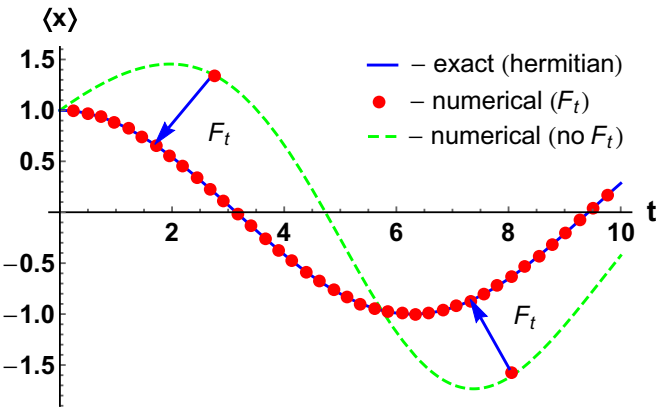

FIG. 2. The average position $\langle x\rangle$ of a quantum particle as a function of time t. The blue solid curve is the exact solution to the Schrödinger equation obtained in the Hermitian representation. Red points represent a numerical solution to the same problem computed in the non-Hermitian (time-dependent) representation. The two paths coincide only if the inertial force is accounted for [see Eq. (18)]. Finally, the green dashed line depicts a "naive" solution obtained without taking into account this contribution. Parameters are $\hbar = m = 1.0$, $\omega = 0.5$, and $\eta_t = t/\tau$, where $\tau = 10$. The initial state is $|\psi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, where $|0\rangle$ is the ground state of $K = H_0$ and $|1\rangle = a^\dagger|0\rangle$.

VI. SUMMARY

The very question whether physical observables should be Hermitian or not reflects on a long-lasting debate regarding physical reality. Nowadays, this issue is no longer only of academic interest as leading groups are beginning to investigate it experimentally [5,8]. In this article we have revisited this problem showing that the repeatability of measurements does not exclude non-Hermitian operators from the usual framework of quantum mechanics. We have argued that operators which admit real spectra are canonically equivalent to Hermitian ones. As a result, all fundamental notions (e.g., repeatability of a measurement, no cloning theorem, etc.) that have been associated with unitarity apply to all non-Hermitian systems with real spectra as well.

The question which of these two representations is more adequate to describe a quantum system depends on the problem under investigation. It may be more natural to use a non-Hermitian frame of reference. However in that case, as a result of using a nonstandard representation, the resulting Schrödinger equation has to be modified accordingly [see Eq. (13)]. There is an extra energetic contribution that has to be accounted for to preserve unitarity. We have associated this energetic cost F_t with an inertial force (quantum Coriolis force) that keeps a quantum system on the unitary path during its evolution (see Sec. V C). As it does in classical mechanics, F_t vanishes for all nonaccelerating frames of reference, i.e., with $\dot{g}_t = 0$.

We should stress here that not all non-Hermitian systems have real spectra. Those whose eigenenergies (at least some of them) are complex were explicitly excluded from our considerations. Such systems are *open* [47]. During their evolution they lose or gain energy and information in a way that cannot be balanced [4]. Therefore, a unitary map is not sufficient to capture their dynamics anymore. Interesting examples can be found, e.g., in Refs. [8,11].

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