Exact reduced dynamics for a qubit in a precessing magnetic field and in contact with a heat bath

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The two-level quantum system (qubit) in a precessing magnetic field and in contact with a heat bath is investigated. The exact reduced dynamics for the qubit in question is obtained. We apply the approach based on the block operator matrices theory, in contrast with the standard methods provided by the theory of the open quantum systems. We also present the solution of the Riccati operator equation associated with the Hamiltonian of the system. Next, we study the adiabatic approximation for the system under consideration using quantum fidelity as a way to measure validity of the adiabatic theory. We find that in the weak-coupling domain the standard condition that ensures adiabatic behavior of the spin in the magnetic field also guarantees its adiabatic evolution in the open-system variant of this model. Therefore, we provide the explicit example of the open quantum system that satisfies the adiabatic approximation first formulated for the closed quantum systems.

I. INTRODUCTION

The most established and useful time-dependent two-level quantum system is, perhaps, the one that describes a spin-half particle (qubit Q) in a precessing magnetic field [1]. The Hamiltonian of this system in its basic variant is usually written in the following form:

$$H_Q(t) = -\gamma \mathbf{S} \cdot \mathbf{B}(t), \quad \mathbf{S} = \boldsymbol{\sigma}/2, \qquad (1)$$

where γ is a constant, called the gyromagnetic ratio (the specific value of γ is irrelevant in context of our discussion), and $\boldsymbol{\sigma} \equiv (\sigma^x, \sigma^y, \sigma^z)$ and σ^i ($i = x, y, z$) are the standard Pauli matrices. The magnetic field $\mathbf{B}(t)$ is assumed to have the form

$$\mathbf{B}(t) = B_1 \cos(\omega t)\hat{e}_x + B_1 \sin(\omega t)\hat{e}_y + B_0 \hat{e}_z, \qquad (2)$$

where $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$ are the unit vectors along x, y, and z axis, respectively. B_0, B_1 define the amplitudes of the magnetic field and ω is the frequency of the rotation. Using the equations above one can also rewrite the Hamiltonian (1) in a more readable form:

$$H_Q(t, \beta) = \beta \sigma_3 + \alpha[\sigma_2 \sin(\omega t) + \sigma_1 \cos(\omega t)], \qquad (3)$$

where for the sake of convenience the abbreviations $\alpha = \frac{1}{2}\omega_1 \equiv -\frac{1}{2}\gamma B_1$ and $\beta = \frac{1}{2}\omega_0 \equiv -\frac{1}{2}\gamma B_0$ were introduced.

In the case of no coupling with the external environment (heat bath) the exact form of the evolution operator $U_Q(t)$ and hence the density matrix $\rho_Q(t) = U_Q(t)\rho_Q U_Q(t)^\dagger$ for the model (3), can be derived in an elegant and simple manner (see, e.g., Refs. [1,2]). This problem is so common that it can be found in almost every modern textbook on quantum mechanics. However, if the aforementioned coupling is present, the *exact* form of the density matrix has not yet been derived.

If one allows the system to interact (not necessarily to exchange the energy) with the environment then it becomes a so-called *open quantum system* [3]. Its time evolution is *not* unitary any more because of the decoherence process [4,5]. Nevertheless, it may always be described by trace-preserving (TP) and completely positive (CP) quantum operation (or quantum channel; see, e.g., Refs. [6]) $T_t : \rho_Q \to \rho_Q(t)$,

$\rho_Q(0) \equiv \rho_Q$. Unfortunately, finding its exact form is almost impossible in most cases, especially for the systems described by the time-dependent Hamiltonian.

Naturally, the question arises: How difficult is this problem? Recently, the efforts to answer this question were made [7,8]. It was shown that from the mathematical perspective this task is at least as complicated as the problem of solving the Riccati algebraic equation associated with the Hamiltonian defining the model in question. Moreover, from the block operator matrices theory point of view obtaining the exact reduced dynamics for any system where decoherence is present (beyond dephasing phenomenon) [9,10] is as complex as solving this equation. Of course, the complexity of the analysis strongly depends on the particular choice of the environment responsible for the decoherence process.

First, we show that all the difficulties mentioned above, including the problem of solving the Riccati equation, can be overcome for the so-called spin environment. Therefore, we provide the exact reduced dynamics for the open-system version of the model (1) in the case when the system is immersed within the spin bath. Next, as an application, we investigate the adiabatic approximation for the open system constructed in this way. Moreover, we show that in the weak-coupling domain the condition ensuring the adiabatic evolution of the system (1) also leads to the adiabatic behavior of its generalization.

The layout of this article is as follows. In Sec. II we briefly review the concept of the block operator matrix and the procedure allowing one to diagonalize it. In Sec. II A we introduce the Riccati operator equation and discuss its connection with the Hamiltonian specifying the model. In Sec. III the definition of the model is given and the exact reduced dynamics is derived. We also indicate possible applications of the model. In Sec. V we investigate the adiabatic approximation. Finally, Sec. VI contains the final remarks and the summary of the article.

II. BLOCK OPERATOR MATRICES APPROACH

A. Partial trace and reduced dynamics

Before we can go any further and present the main result of this article let us first discuss the procedure allowing one to

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obtain the density matrix $\rho_Q(t)$ for a two-dimensional system (qubit) using the block operator matrices approach.

Let H_{QE} be the Hamiltonian of the closed (qubit and environment, $Q + E$) system. We will assume its following form

$$H_{QE} = H_Q \otimes \mathbb{I}_E + \mathbb{I}_Q \otimes H_E + H_{\text{int}}, \qquad (4)$$

where H_Q, H_E are the Hamiltonian of the qubit and the environment, respectively, while H_{int} represents the interaction between the systems. Hamiltonian (4) acts on $\mathcal{H}_{QE} = \mathcal{H}_Q \otimes \mathcal{H}_E$ space, where $\mathcal{H}_Q$ and $\mathcal{H}_E$ are the Hilbert spaces for the qubit and the environment, respectively. The state $\rho_Q(t)$ of the qubit at any given time t can be computed using the following formula (for more details see, e.g., Ref. [11])

$$\begin{aligned} \rho_Q(t) &= \text{Tr}_E(U_t \rho_Q \otimes \rho_E U_t^\dagger) \\ &\equiv T_t(\rho_Q), \end{aligned} \qquad (5)$$

where $U_t = \exp(-i H_{QE} t)$ is the evolution operator of the total $Q + E$ system (we work with units $\hbar = 1$). By $\text{Tr}_E(\cdot)$ we denoted the partial trace. In the literature the quantity $\rho_Q(t)$ is commonly called the *reduced dynamics*. In this article we will be also referring to it as the solution of a given model. In Eq. (5) the assumption that $\rho_{QE} = \rho_Q \otimes \rho_E$ was made, i.e., no correlation between Q and E are present initially (see, e.g., Refs. [12,13] and references therein).

Since $\mathcal{H}_Q = \mathbb{C}^2$ and the isomorphism $\mathbb{C}^2 \otimes \mathcal{H}_E = \mathcal{H}_E \oplus \mathcal{H}_E$ holds, one can think of the operator H_{QE} (and of any given operator A acting on $\mathcal{H}_Q \otimes \mathcal{H}_E$) as a 2×2 block operator matrix [14]. Let us assume that A acts on $\mathcal{H}_E \oplus \mathcal{H}_E$. Thus, we can write $A = [A_{ij}]$, where operators A_{ij}, $(i, j = 1, 2)$ act on $\mathcal{H}_E$. In this block operator matrices nomenclature the procedure of calculating the partial trace $\text{Tr}_E(\cdot)$ can be written as

$$\text{Tr}_E(A) = \begin{pmatrix} \text{Tr} A_{11} & \text{Tr} A_{12} \\ \text{Tr} A_{21} & \text{Tr} A_{22} \end{pmatrix}, \qquad (6)$$

where $\text{Tr}(\cdot)$ denotes the trace on $\mathcal{H}_Q$. Equation (6) introduces the concept of the partial trace in a very simple and, most importantly, intuitive way. It is also possible to define the partial trace locally, i.e., $\text{Tr}_E(X \otimes Y) = X \text{Tr}(Y)$, where, as mentioned, $\text{Tr}(\cdot)$ stands for the trace operation on the space on which the operator Y acts. From Eq. (6) one can learn that the partial trace is the operation which transforms operator matrices into the "ordinary" matrices. Moreover, the partial trace is a linear operation going from $B(\mathcal{H}_{QE})$ space to the $B(\mathbb{C}^2)$ space. Furthermore, from the block operator matrices perspective one can easily understand why dealing with the open systems is much more complex (from the mathematical viewpoint) than dealing with the closed systems. The reason is that the procedure that allows one to calculate the reduced dynamics is not as straightforward as Eq. (6) might indicate. Indeed, to compute the partial trace (5) one needs to write the evolution operator U_t in a 2×2 block matrix form. The last operation requires diagonalization of the block operator matrix H_{QE}, which is much more complicated than the diagonalization of the standard 2×2 matrix. It leads to the problem with solving the Riccati algebraic equation analyzed below.

B. Riccati equation and block diagonalization

With every Hermitian block operator matrix of the form

$$R = \begin{bmatrix} A & B \\ B^\dagger & C \end{bmatrix}, \qquad (7)$$

where A, B, and C are the Hermitian operators acting on $\mathcal{H}_E$, one can associate the following Riccati equation [15]

$$XBX + XA - CX - B^\dagger = 0, \qquad (8)$$

where $X \in B(\mathcal{H}_E)$ is the operator to be determined.

If the solution X of Eq. (8) exists, then it may be used to diagonalize operator matrix (7). The procedure that allows us to do so is as follows (for details see Ref. [8] and references therein)

$$U_X^{-1} R U_X = \begin{bmatrix} A + BX & 0_E \\ 0_E & C - (XB)^\dagger \end{bmatrix}, \qquad (9)$$

where the similarity matrix U_X is given by

$$U_X = \begin{bmatrix} \mathbb{I}_E & -X^\dagger \\ X & \mathbb{I}_E \end{bmatrix}. \qquad (10)$$

We wish to emphasize that in most cases a general method of finding the solution for the Riccati equation does not exist. Therefore, solving the Eq. (8) with respect to X is a very difficult task. Moreover, even if the solution is known, there is still a problem of computing the inversed operator matrix $(U_X)^{-1}$. Formally, if the operator X is normal, i.e., $[X, X^\dagger] = 0$, then this matrix is given by the formula

$$U_X^{-1} = (\mathbb{I}_E + XX^\dagger)^{-1} \begin{bmatrix} \mathbb{I}_E & X^\dagger \\ -X & \mathbb{I}_E \end{bmatrix}. \qquad (11)$$

Otherwise, the form of the inverse operator U_X^{-1} becomes more complicated. Fortunately, Eq. (11) is sufficient for our analysis. Note that it may be difficult to handle computationally the operator such as $(\mathbb{I}_E + XX^\dagger)^{-1}$. In general the operator X does not need to be Hermitian or unitary.

Perhaps those are the reasons why the approach to open systems based on the operator matrices theory has not attracted too much attention. In our opinion, however, the operator matrix perspective can offers a better understanding of the open quantum systems and might give some new results in this area.

III. EXACT REDUCED DYNAMICS

A. Model

We consider the model defined by the following time-dependent Hamiltonian

$$H_{QE}(t, \beta) = H_Q(t, \beta) \otimes \mathbb{I}_E + \mathbb{I}_Q \otimes H_E + H_{\text{int}}, \qquad (12)$$

where $H_Q(t, \beta)$ is given by Eq. (3) and it represents a qubit in the magnetic field (2). The environment is composed of N independent and noninteracting spin-half particles. The Hamiltonian H_E of the bath is assumed to be of the form

$$H_E = \sum_{n=1}^{N} \omega_n \sigma_n^z, \qquad (13)$$

where ω_n and σ_n^z, $(n = 1,...,N)$ are the frequencies and the Pauli matrices for nth qubit, respectively. The Hilbert space $\mathcal{H}_E$ on which the Hamiltonian (13) acts is given by N-fold tensor product of $\mathbb{C}^2$ spaces, i.e., $\mathcal{H} = \bigotimes_{n=1}^{N} \mathbb{C}^2$. Therefore for any $n \leqslant N$ the operator σ_n^z is understood as $\sigma_n^z = I_2 \otimes \ldots \otimes \sigma^z \otimes \ldots \otimes I_2$, where σ^z is the standard Pauli matrix acting on $\mathbb{C}^2$ and I_2 is the 2×2 identity matrix on that space. In our model the coupling of a qubit with the environment is governed by the Ising-typ Hamiltonian [16,17], namely

$$H_{\text{int}} = \sigma^z \otimes \sum_{n=1}^{N} g_n \sigma_n^z, \tag{14}$$

where g_n are the coupling constants. We assume that the bath is initially in the Gibbs thermal state at a temperature T, i.e.,

$$\rho_E = Z^{-1} \exp(-H_E/kT), \tag{15}$$

where $Z = \text{Tr}[\exp(-H_E/kT)]$ and k is the Boltzmann constant.

The model described by the Hamiltonian (12) with H_E and V specified by Eqs. (13)–(14) in the case of the $\alpha = 0$ (i.e., dephasing and static magnetic field case) was investigated both in the context of the approximation methods in the open-quantum-systems theory [16] and the capacities of the quantum channels [17]. Thus, for the detailed discussion and possible applications we refer the reader therein.

The quantum system discussed here can be thought of as the generalization of the model introduced in the mentioned articles to the case when the energy is exchanged between the systems and the magnetic field depends on time. It is worth mentioning the physical problems that make use of this model. First, the Hamiltonian (12) may pose a useful prototype describing any spin "resonance" phenomenon, like, for instance, nuclear magnetic resonance (NMR) [18,19]. In such a picture, the spin-half particle is the open system, and we wish to describe its evolution in time. The spin bath models the influence of the other spins on the open system in question. Finally, the rotating magnetic field is used to induce the resonance and to control the system.

The quantum devices of the future, like, for instance, quantum computers [20], will be probably built from the components that are composed of a large amount of the qubits (spins, in particular), e.g., quantum memory [21], quantum register [22], and so on. The model we investigate may serves as a simple prototype that allows one to trace the evolution in time of a single qubit of the register or memory. The magnetic field may be then applied to program the device. Those are just a few potential applications of the simple theoretical, time-dependent spin-spins model (12).

B. Exact solution

To derive an exact reduced dynamics of the system governed by the Hamiltonian (12) we first simplify this problem to a time-independent one. In order to accomplish this, we use a recently proven theorem that says if (for more details see Appendix A and also Ref. [8]) (i) the total system is defined by the Hamiltonian (12); (ii) the interaction term

H_{int} between the systems takes the form $f(\sigma^z) \otimes V$, where f is an analytical function of σ^z and V is a Hermitian operator; and (iii) $\rho_t(\beta)$ represents the solution of the system defined by the Hamiltonian $H_{QE}(0,\beta)$ (note that Hamiltonian H_{QE} depends on β, hence so is ρ_t), then the reduced dynamics η_t for the model governed by the Hamiltonian (12) can be obtained using the following simple formula:

$$\eta_t = V_t \rho_t(\beta_{\text{eff}}) V_t^{\dagger}, \quad \beta_{\text{eff}} := \beta - \omega/2, \tag{16}$$

where the unitary matrix V_t is given by (diag-diagonal)

$$V_t = \text{diag}(e^{-i\omega t/2}, e^{i\omega t/2}). \tag{17}$$

Equality (16) states that having the reduced dynamics ρ_t one may easily obtain the solution we are interested in simply by introducing the *effective* parameter β_{eff}, then replacing β by β_{eff}, and finally performing the unitary transformation (17).

The interaction term defined in Eq. (14) satisfies the requirement of the theorem above, thus one should restrict the analysis to the Hamiltonian $H_{QE}(\beta) \equiv H_{QE}(0,\beta)$. The latter can be easily rewritten as a 2×2 block operator matrix, namely

$$H_{QE}(\beta) = \begin{bmatrix} H_+(\beta) & \alpha\mathbb{I}_E \\ \alpha\mathbb{I}_E & H_-(\beta) \end{bmatrix}, \tag{18}$$

where

$$H_{\pm}(\beta) := \sum_{n=1}^{N} \left(\omega_n^{\pm}\sigma_n^z \pm \bar{\beta}\mathbb{I}_E \right), \quad \bar{\beta} := \beta/N, \tag{19}$$

and $\omega_n^{\pm} := \omega_n \pm g_n$, for each $n \leqslant N$. Henceforward, the explicit dependence of any quantity of parameter β will be omitted until the solution $\rho_Q(t)$ is obtained.

The evolution operator for the total system generated by the Hamiltonian (18) reads

$$\begin{aligned} U_t &= \exp(-iH_{QE}t) \\ &= U_X \exp(-iH_d)U_X^{-1}, \end{aligned} \tag{20}$$

where U_X is given by Eq. (10) and X is the solution of the Riccati Eq. (8), which in the present case takes the form

$$\alpha X^2 + XH_+ - H_-X - \alpha\mathbb{I}_E = 0. \tag{21}$$

The quantity H_d represents a diagonal form of the operator matrix H_{QE}. According to Eq. (9) it reads

$$H_d = \begin{bmatrix} H_+ + \alpha X & 0_E \\ 0_E & H_- - \alpha X^{\dagger} \end{bmatrix}. \tag{22}$$

Note that for $\alpha = 0$ the solution (possibly not the only one) of Eq. (21) is given by $X_0 = 0_E$. It is obvious, since in that case H_{QE} is already in the diagonal form, i.e., $H_{QE} = H_d = \text{diag}(H_+, H_-)$. In order to obtain the solution for $\alpha \neq 0$ more subtle investigation, provided in the next subsection, is needed.

1. Solution of the Riccati equation for the α ≠ 0 case

First, one can observe that $[H_-, H_+] = 0$. Since the solution X of Eq. (21) is a function of the operators $H_{\pm}$, then it also

must commute with those operators. The Riccati equation in question can be simplified to the more compact form, namely

$$\alpha X^2 + 2VX - \alpha \mathbb{I}_E = 0, \quad \alpha \neq 0, \tag{23}$$

where we introduced

$$V = \sum_{n=1}^{N} \left(g_n \sigma_n^z + \bar{\beta} \mathbb{I}_E \right). \tag{24}$$

Let us assume that the eigenvalue problem for the operator V can be (easily) resolved. If this is the case we can write the solution X of Eq. (23) in a manageable form using the spectral theorem for the Hermitian operators. By doing so we obtain that $X = f(V)$, where

$$f(\lambda) = \frac{\sqrt{\lambda^2 + \alpha^2} - \lambda}{\alpha}, \quad \lambda \in \sigma(V). \tag{25}$$

We can also represent the solution $f(V)$ in an equivalent way, i.e.,

$$f(V) = \sum_{\lambda \in \sigma(V)} f(\lambda)|\lambda\rangle\langle\lambda|. \tag{26}$$

Note that $f = f^*$, i.e., f is a real function for any value of the parameters α and β. Here, by $\sigma(V)$ we denoted the spectrum of V. We want to emphasize that the representation (25) or (26) of the solution X of Eq. (23) is useful computationally if the eigenvalues of V can be computed. We show below that this is the case. Note also that $X = X^\dagger$ if and only if $V = V^\dagger$.

Let $i = i_1 i_2 ... i_N$ be a binary ($i_n = 0,1$, for $n \leqslant N$) expansion of an integer number $i \in [0, 2^N - 1]$. Clearly, the set of all eigenstates $|i\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_N\rangle$ forms an orthonormal basis in $\mathcal{H}_E$. Moreover, because of the following equality $\sigma^z|i_n\rangle = (-1)^{i_n}|i_n\rangle$ we have

$$V|i\rangle = \sum_{n=1}^{N} [g_n(-1)^{i_n} + \bar{\beta}]|i\rangle$$
$$\equiv E_i|i\rangle, \quad 0 \leqslant i \leqslant 2^N - 1. \tag{27}$$

Thus, the eigenvalue problem for V has been solved. Therefore, the solution of the Riccati Eq. (23) X reads

$$X = \sum_{i=0}^{2^N-1} f(E_i)|i\rangle\langle i|. \tag{28}$$

At this point two remarks should be made. The first one is that the operator (28) depends on the parameter α, i.e., $X = X(\alpha)$ and $X(\alpha) \to X_0$ as α goes to 0. To see that this statement holds we rewrite Eq. (25) in the form

$$f(\lambda) = \frac{1}{\sqrt{\left(\frac{\lambda}{\alpha}\right)^2 + 1} + \frac{\lambda}{\alpha}}, \quad \lambda \in \sigma(V). \tag{29}$$

Clearly, $f(\lambda) \to 0$ as $\alpha \to 0$, unless $\lambda = 0$. The latter means that the solution X of the Riccati Eq. (23) is a continuous function of the parameter α, including the $\alpha = 0$ value. From the physical point of view it means that one can control energy exchange between the systems in a continuous way. Note that

if $\alpha = 0$, then the energy transfer is absent. This scenario can be accomplished by taking limits $\alpha \to 0$ with the final solution we are about to give. Note also that since for every $\alpha \neq 0$ and each $\lambda \in \sigma(V)$ the eigenvalues $f(\lambda)$ of the operator X we constructed are positive. This solution is a positively defined operator on $\mathcal{H}_E$.

The second remark is that there exists, at least one more operator function that satisfies Eq. (23), namely the one specified by the following formula: $f_2(\lambda) = (-\sqrt{\lambda^2 + \alpha^2} - \lambda)/\alpha$ for $\lambda \in \sigma(V)$. This solution represents negatively defined operator. The choice between the different solution of Eq. (23) is *not* arbitrary and it has significant influence on the further analysis. Indeed, if one decides to choose the second solution: $f_2(\lambda)$ then one may meet a serious difficulties during the examination of the dephasing phenomena, since $f_2(\lambda) \to -\infty$ as $\alpha \to 0$.

At the end of this subsection let us note that one can resolve the eigenvalue problem for $H_\pm$ as easily as for the operator V. Indeed, we may readily verify that

$$H_\pm|i\rangle = \sum_{n=1}^{N}[\omega_n^\pm(-1)^{i_n} \pm \bar{\beta}]|i\rangle$$
$$\equiv E_i^\pm|i\rangle, \quad 0 \leqslant i \leqslant 2^N - 1. \tag{30}$$

Observe also that $E_i^+ - E_i^- = 2E_i$, for $0 \leqslant i \leqslant 2^N - 1$, as one may expected. In the $|i\rangle$ basis the density matrix ρ_E can be expanded as

$$\rho_E = \sum_{i=0}^{2^N-1} (e^{-\Omega_i/kT}/Z)|i\rangle\langle i|, \tag{31}$$

where $\Omega_i = \sum_n \omega_n(-1)^{i_n}$ are the eigenvalues of H_E and $Z = \sum_i \exp(-\Omega_i/kT)$.

2. Total system evolution

We are now in a position to give the explicit and manageable form of the evolution operator of the total system $Q + E$. First, let us note that the similarity operator matrix U_X can be written as

$$U_X = \sum_{i=0}^{2^N-1} U_i \otimes |i\rangle\langle i|, \tag{32}$$

where matrices U_i are given by

$$U_i = \begin{pmatrix} 1 & -f(E_i) \\ f(E_i) & 1 \end{pmatrix}. \tag{33}$$

This follows immediately from Eqs. (10) and (28). We also used the resolution of the identity $\mathbb{I}_E$ in the $|i\rangle$ basis. From the expansion (32) one can readily see that the inversed operator $(U_X)^{-1}$ takes the form

$$(U_X)^{-1} = \sum_{i=0}^{2^N-1} (U_i)^{-1} \otimes |i\rangle\langle i|, \tag{34}$$

where $(U_i)^{-1}$ stands for the inverse of the matrix U_i. Since for every integer i we have $\det(U_i) = 1 + f^2(E_i) \neq 0$, thus the inverse exists. It is also easy to see that $(U_i)^{-1} = \det(U_i)U_i^\dagger$

and $U_i U_i^\dagger = 1/\det(U_i)$. Because of the last equality, the operator (32) can become unitary by a simple rescaling procedure, i.e., $U_i \to \sqrt{\det U_i} U_i$. Furthermore, according to Eq. (20) the diagonal form $U_t^d := \exp(-i H_d t)$ reads

$$U_t^d = \sum_{i=0}^{2^N-1} U_i^d(t) \otimes |i\rangle\langle i|, \qquad (35)$$

where abbreviations

$$U_i^d(t) = \mathrm{diag}(e^{-i[E_i^+ + \alpha f(E_i)]t}, e^{-i[E_i^- - \alpha f(E_i)]t}), \qquad (36)$$

were introduced.

Finally, by combining all the results together we obtain the evolution operator U_t of the total system:

$$U_t = \sum_{i=0}^{2^N-1} U_i(t) \otimes |i\rangle\langle i|, \qquad (37)$$

where

$$
\begin{aligned}
U_i(t) &= U_i U_i^d(t) U_i^\dagger \\
&= \frac{1}{1+f_i^2} \begin{pmatrix} e_i^+(t) + e_i^-(t) f_i^2 & f_i[e_i^+(t) - e_i^-(t)] \\ f_i[e_i^+(t) - e_i^-(t)] & e_i^+(t) f_i^2 + e_i^-(t) \end{pmatrix}.
\end{aligned}
\qquad (38)
$$

and $e_i^\pm(t) := \exp[-i(E_i^\pm \pm \alpha f_i)t]$ with $f_i \equiv f(E_i)$. Let us keep in mind that $U_i U_i^\dagger = \mathbb{I}_Q$ for each integer i.

3. Reduced dynamics

We begin with the assumption that ρ_Q and ρ_E are arbitrary density operators for the open system and its environment, respectively. Furthermore, we consider only factorable initial states ρ_{QE} of the closed system $Q + E$, i.e., $\rho_{QE} = \rho_Q \otimes \rho_E$. Using this assumption and the explicit form of the evolution operator U_t given in Eq. (37), we have that the density operator $\rho_{QE}(t) = U_t \rho_{QE} U_t^\dagger$ at any given time reads

$$\rho_{QE}(t) = \sum_{i,j=0}^{2^N-1} U_i(t)\rho_Q U_j(t)^\dagger \otimes |i\rangle\langle i|\rho_E|j\rangle\langle j|. \qquad (39)$$

Tracing out the last equation over the environment degrees of freedom, we obtain the reduced dynamics [note $\rho_Q(t) = \mathrm{Tr}_E(\rho_{QE}(t))$]

$$\rho_Q(t) = \sum_{i=0}^{2^N-1} \rho_i U_i(t)\rho_Q U_i(t)^\dagger, \qquad (40)$$

where $\rho_i \equiv \langle i|\rho_E|i\rangle$. By introducing matrices $K_{ij}(t)$ such that $K_{ij}(t) := \delta_{ij}\sqrt{\rho_i} U_i(t)$, the equation above can be written in a more familiar form, that is $\rho_Q(t) = \sum_{ij} K_{ij}(t)\rho_Q K_{ij}(t)^\dagger$. Moreover, it is easily to verify that $\sum_{ij} K_{ij}(t)K_{ij}(t)^\dagger = \mathbb{I}_Q$. Thus, Eq. (40) can be thought of as operator sum representation of the state $\rho_Q(t)$. The matrices $K_{ij}(t)$ are the Kraus operators. The result (40) is general, i.e., it holds for the arbitrary states ρ_Q and ρ_E. Nevertheless, we made assumption about the form of the initial state ρ_{QE}. This assumption, however, is not essential and does not lead to the limitation of the analysis. The evolution

operator (37) was written in a highly manageable form, and by that we mean it can be applied to *any* given initial state. Of course, if initial correlations are present, then Eqs. (39) and (40) no longer hold, but the reduced dynamics $\rho_Q(t)$ can still easily be obtained. However, we will not focus on this issue in the current article.

So far, we omitted the explicit dependency on β in the introduced quantities. Note, however, that if $U_i(t) = U_i(t,\beta)$, then $\rho_Q(t) = \rho_Q(t,\beta)$ and, in view of Eqs. (16), (40), and (17), and comments following them, we have

$$\eta_t = \sum_{i,j=0}^{2^N-1} M_{ij}(t)\eta_Q M_{ij}(t)^\dagger, \qquad (41)$$

where η_Q ($\eta_Q = \rho_Q$) is an arbitrary initial state. The Kraus matrices $M_{ij}(t)$ are given by

$$M_{ij}(t) = \delta_{ij}\sqrt{\rho_i} V_t U_i(t,\beta_{\mathrm{eff}}). \qquad (42)$$

The last two equations are the main result of the current article. It is worth mentioning that the model with a similar properties was studied in Ref. [23], where the authors obtained the operator sum representation, yet the Kraus operator introduced therein involved the time chronological operator.

From Eqs. (41) and (42) one can learn that the time evolution of the qubit interacting with the rotating magnetic field (2) and in contact with the environment (13)–(14) can be described using two quantum channels, i.e., $\rho_Q(t) = T_t^2 \circ T_t^1(\rho_Q)$. The first channel, that is, T_t^1, is given by

$$T_t^1 := \sum_{i=0}^{2^N-1} M_i(t)(\cdot)M_i(t)^\dagger. \qquad (43)$$

The second one is defined by the simple unitary operation, namely $T_t^2 = V_t(\cdot)V_t^\dagger$. The matrices M_i are specified by

$$M_i(t) = \sqrt{\rho_i}\exp(-i H_i t), \qquad (44)$$

where the Hamiltonian H_i is the generator of the unitary evolution $U_i(t,\beta_{\mathrm{eff}})$ [see (38)]. It is easy to compute that

$$H_i = \begin{pmatrix} E_i - \omega/2 & \alpha \\ \alpha & -E_i + \omega/2 \end{pmatrix}, \qquad (45)$$

where E_i is the eigenvalues of the operator V and it was defined in Eq. (27). Channel T_t^2 may be thought of as the convex combination of the unitary channels [note $T_t(\mathbb{I}_Q) = \mathbb{I}_Q$]. These types of channels are known as the *random unitary channels* [24] and they often appear in the study of pure decoherence.

Surprisingly, the matrix above and therefore the Kraus matrices (44) does *not* depend on the function $f(E_i)$. However, to compute matrices $\exp(-i H_i t)$ that specified the Kraus matrices (44) we need to diagonalize the Hamiltonians H_i. The function $f(E_i)$ is hidden in the similarity matrix U_i given by Eq. (33). Observe that the mentioned diagonalization procedure is based on the Riccati equation (matrices U_i are composed with the eigenvalues of the solution of the Riccati equation). This approach differs from the standard method based on the characteristic equation. This is a new kind of the

diagonalization procedure, so-called Riccati diagonalization [25].

IV. SPECIAL CASES

Now, we look into some common cases that may arise during the examination of the model (12). We will show that in these particular situations the result (41) can be simplified to the well-known expressions.

A. No coupling with the bath

In the case when the qubit under consideration has no coupling with its environment, i.e., $g_n = 0$ for all $n \leqslant N$ we have (a) $E_i = \beta_{\text{eff}}$, hence and (b) $f_i \equiv f = (|z| - \beta_{\text{eff}})/\alpha$, where (c) $z := \alpha + i\beta_{\text{eff}}$. Moreover, $E_i^{\pm} = \Omega_i \pm \beta$, where $\Omega_i \equiv \sum_n \omega_n (-1)^{i_n}$. Using this simplification we obtain that $U_i(t, \beta_{\text{eff}}) = e^{-i\Omega_i t} \bar{U}(t)$, where

$$
\bar{U}(t) = \begin{pmatrix} e^{-i|z|t} + f^2 e^{i|z|t} & -2if \sin(|z|t) \\ -2if \sin(|z|t) & f^2 e^{-i|z|t} + e^{i|z|t} \end{pmatrix}
$$

$$
= I_2 \cos(|z|t) - i \begin{pmatrix} \frac{1-f^2}{1+f^2} & \frac{2f}{1+f^2} \\ \frac{2f}{1+f^2} & -\frac{(1-f^2)}{1+f^2} \end{pmatrix} \sin(|z|t)
$$

$$
= I_2 \cos(|z|t) - i\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\Omega}} \sin(|z|t)
$$

$$
= \exp(-i|z|\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\Omega}} t), \tag{46}
$$

with $\hat{\boldsymbol{\Omega}} := \boldsymbol{\Omega}/\|\boldsymbol{\Omega}\|$ and $\boldsymbol{\Omega} := (2f, 0, 1 - f^2)$. As a result, $M_{ij}(t) = \delta_{ij} \sqrt{\rho_i} V_t \bar{U}(t)$, where we dropped phase factor $e^{-i\Omega_i t}$. Therefore $\eta_t = U(t)\eta U(t)^{\dagger}$ (as one may anticipate), where

$$
U(t) = \exp(-i\omega t \sigma^z/2) \exp(-i|z|\boldsymbol{\sigma} \cdot \hat{\boldsymbol{\Omega}} t). \tag{47}
$$

This is the well-known formula for the (unitary) evolution operator of the system identified by the Hamiltonian $H_Q(t, \beta) \equiv H_Q(t)$. It is the solution of the differential equation $i\partial_t U(t) = H_Q(t)U(t)$. Note, that the result (47) corresponds to the Floquet decomposition of the unitary operator generated by the periodic Hamiltonian. Note also that the evolution operator (47) can be written in a more readable form, especially useful for studying the adiabatic approximation (see Sec. V), i.e.,

$$
U(t) = V_t \exp(-iHt), \tag{48}
$$

where the matrix V_t is given by Eq. (17) and H has the form

$$
H = \begin{pmatrix} \beta_{\text{eff}} & \alpha \\ \alpha & -\beta_{\text{eff}} \end{pmatrix}. \tag{49}
$$

One can easily see that this matrix can be obtained directly from Eq. (45) if one sets $g_n = 0$ for $n \leqslant N$.

B. Dephasing

If the constant α equals zero, then the Hamiltonian $H_{QE}(t, \beta)$ becomes time independent. Moreover, in this circumstance $[H_Q, H_{\text{int}}] = 0$, hence the open system does not exchange energy with its environment, i.e., the pure decoherence occurs. It poses no problems to check that $U_i(t, \beta_{\text{eff}}) = e^{-i\Omega_i t} V_t^{\dagger} \exp(-iE_i \sigma^z t)$. Therefore, the Kraus operators take the form $M_{ij}(t) = \delta_{ij} \sqrt{\rho_i} \exp(-iE_i \sigma^z t)$. Just like before the

phase factor $e^{-i\Omega_i t}$ was dropped. In agreement with (41), the form above of the Kraus matrices leads to the following manifestation of the density matrix η_t

$$
\eta_{11}(t) = \eta_{11}, \quad \eta_{12}(t) = \sum_{i=0}^{2^N - 1} \rho_i e^{-i2E_i t} \eta_{12}. \tag{50}
$$

Naturally, $\eta_{22}(t) = 1 - \eta_{11}(t)$ and $\eta_{21}(t) = \eta_{21}(t)^*$. Note that the coherence $C(t) := |\eta_{12}(t)|$ does not decay exponentially (or anyhow for finite N) in the long time regime but manifests oscillating behavior.

From the considerations above one can easily learn that there is a substantial difference between the situation when a transfer of the energy from the qubit to its bath is not present and the case when the qubit is not coupled to the environment. Let us also observe that in the model we study it is not possible to construct a situation when the energy exchange in not present ($\alpha = 0$) and yet the magnetic field is rotating. This follows from the fact that if $\alpha = 0$ then the Hamiltonian $H_{QE}(t, \beta)$ does not depend on ω.

C. Low-temperature regime

From the mathematical point of view the zero- (low-) temperature limit is the most subtle special case. This comes from the fact that most results obtained by using the known approximation methods in the open-quantum-systems theory, e.g., the Markovian or the singular coupling limit, cannot be extrapolated to this regime. Luckily, the exactly solvable models are unhampered by these difficulties.

To derive the exact reduced dynamics in the low-temperature domain we begin with rewriting the density operator ρ_E in a more suitable form in contrary with the one given by Eq. (31). According to the result obtained in Ref. [16] one may write

$$
\rho_E = \bigotimes_{n=1}^{N} \frac{1}{2}(\mathbb{I}_Q + \beta_n \sigma^z), \tag{51}
$$

where $\beta_n = \tanh(-\omega_n/kT)$. From equality (51) one may readily see when $T \to 0$, then

$$
\rho_i = \frac{1}{2^N} \prod_{n=1}^{N} [1 + \beta_n (-1)^{i_n}] \to \delta_{i0}, \tag{52}
$$

since $\beta_n \to 1$ as $T \to 0$. All this means that in the low-temperature domain the heat bath is its ground state, i.e., $\rho_E = |0_E\rangle\langle 0_E|$. As an immediate result we obtain the following form of the reduced dynamics (41)

$$
\lim_{T \to 0} \eta_t = W_t \eta W_t^{\dagger}, \tag{53}
$$

where $W_t := V_t U_0(t, \beta_{\text{eff}})$. Interestingly, the evolution of the open system in question is unitary, which is rather unusual. This is a direct consequence of the fact that for every integer i the operator $U_i(t)$ specified by Eq. (38) is unitary. This means that in the low-temperature regime the dissipative properties of the heat bath (13)–(14) are frozen.

D. Equal couplings and frequencies

Henceforward and without essential loss of generality we assume that $\omega_n = \Omega$ and $g_n = g$ for $n \leqslant N$, i.e., couplings constants and the frequencies of the spins of the bath are equal. In this situation $E_i = g(N - 2k) + \beta$, where k is the *Hamming weight* of the integer number i (i.e., a number of nonzero element in a binary expansion of i). Since there are $\binom{N}{k}$ integer number $i \in [0, 2^N - 1]$ with the same Hamming weight k, the channel (43) takes the form

$$T_t^1 = \sum_{k=0}^{N} \binom{N}{k} M_k(t)(\cdot) M_k(t)^\dagger, \qquad (54)$$

with $M_k(t)$ given by Eq. (44). Note, in this case ρ_k [see Eq. (51)] simplified to the form

$$\rho_k = \frac{1}{2^N}(1 + \delta)^{N-k}(1 - \delta)^k, \qquad (55)$$

where $\delta := \tanh(-\Omega/kT)$.

V. APPLICATION: ADIABATIC APPROXIMATION

The adiabatic theorem [26,27] for the *closed* quantum system specified by the Hamiltonian $H(t)$ in its basics variant [i.e., discrete and no degenerate spectrum $\sigma(H(t))$] states that if $H(t)$ varies slowly (for rigorous meaning of that see e.g., Refs. [28] and references therein) and if the system is initially prepared in one of the eigenstates of $H(0)$, say $|\psi_n(0)\rangle$, then at any given time t the probability $|\langle \psi_n(t)|\psi(t)\rangle|^2$ of finding it in the eigenstate $|\psi_n(t)\rangle$ [an eigenvector of $H(t)$] is equal to 1. Therefore, one may easily introduce the quantity $F(t)$ that measures the validity of the adiabatic approximation, namely

$$F(t) = \mathrm{Tr}(\rho(t)\rho_\psi(t)), \qquad (56)$$

where $\rho_\psi(t) = |\psi_t\rangle\langle\psi_t|$ and $\rho(t) = U_t\rho(0)U_t^\dagger$ with $\rho(0) = |\psi_0\rangle\langle\psi_0|$. Here, $|\psi_t\rangle$ is the eigenvector of $H(t)$ for $t \geqslant 0$ and U_t represents the (unitary) evolution operator for the closed system. If the system is evolving adiabatically then $F(t) = 1$. Note that definition of the quantity $F(t)$ coincides with the standard definition of the quantum fidelity $F(\rho(t), \rho_\psi(t))$ between *pure* states $\rho(t)$ and $\rho_\psi(t)$ [29]. Since this is a two-dimensional case it also coincides with the definition of the superfidelity [30,31].

Observe that Eq. (56) is very useful since it can be easily applied both to the closed and to the open quantum systems. In the last case we need to replace the equality $\rho(t) = U_t\rho(0)U_t^\dagger$ representing the evolution of the closed system by the channel describing evolution in time of the open system. More precisely, $\rho(t) = T_t(\rho(0))$, where T_t is TP-CP quantum operation. In the current section we use Eq. (56) in order to investigate behavior of the model (12) in the adiabatic regime. To make the article self-contained we briefly discuss the adiabatic approximation for the system (1) first.

A. The closed spin system case

For the sake of simplicity we put $\beta = \beta_0 \cos\varphi$ and $\alpha = \beta_0 \sin\varphi$ for certain φ and β_0. This assumptions lead to the following form of the Hamiltonian (3)

$$H_Q(t) = \beta_0 \begin{pmatrix} \cos\varphi & e^{-i\omega t}\sin\varphi \\ e^{i\omega t}\sin\varphi & -\cos\varphi \end{pmatrix}. \qquad (57)$$

It can be easily verified that the matrix above has the eigenvalues $E_\pm = \pm\beta_0$. The corresponding eigenvectors $|\psi_t^\pm\rangle$ are given by

$$|\psi_t^+\rangle = \begin{pmatrix} \cos(\varphi/2) \\ e^{i\omega t}\sin(\varphi/2) \end{pmatrix}, \quad |\psi_t^-\rangle = \begin{pmatrix} \sin(\varphi/2) \\ -e^{i\omega t}\cos(\varphi/2) \end{pmatrix}. \qquad (58)$$

The density matrix $\rho_+(t) = |\psi_t^+\rangle\langle\psi_t^+|$ takes the form

$$\rho_+(t) = \begin{pmatrix} \cos^2(\varphi/2) & \frac{1}{2}e^{-i\omega t}\sin\varphi \\ \frac{1}{2}e^{i\omega t}\sin\varphi & \sin^2(\varphi/2) \end{pmatrix}. \qquad (59)$$

Observe that $\rho_+(t) = V_t\rho_+(0)V_t^\dagger$, where V_t is specified in Eq. (17). According to Eq. (48) the density matrix $\rho(t)$ at any given time t reads $\rho(t) = V_t\bar{U}(t)\rho_+(0)(V_t\bar{U}(t))^\dagger$, thus the fidelity (56) takes the form

$$\begin{aligned} F(t) &= \mathrm{Tr}(\bar{U}(t)\rho_+(0)\bar{U}(t)^\dagger\rho_+(0)) \\ &= \|\bar{U}(t)\rho_+(0)\|_F^2, \end{aligned} \qquad (60)$$

where $\|\cdot\|_F$ is the Frobenius (Euclidean) norm, i.e., $\|A\|_F^2 := \mathrm{Tr}(AA^\dagger)$.

It is well known that for the system governed by the Hamiltonian (57) condition that guarantees the adiabatic evolution is $\beta_0 \gg \omega$. This statement is very intuitive; it means that the magnetic field rotates slowly, in comparison with the phase of the state vector (for a more detailed discussion, see, e.g., Ref. [32]). We can also see this from Eq. (60). Indeed, if one introduces the *adiabatic parameter* $x := \omega/2\beta_0$, then, in agreement with (48) and (60), we have

$$F(t) = 1 - \frac{x^2}{1 + x^2}\sin^2(\Omega(x)t), \qquad (61)$$

where $\Omega(x) := \beta_0\sqrt{1 + x^2}$. Without loss of generality we set $\varphi = \pi/2$ in the equation above. The parameter x measures how slowly the magnetic field rotates in β_0 units, thus in the adiabatic limits (i.e., $x \to 0$) the second term in Eq. (61) can be neglected and therefore $F(t) \simeq 1$.

B. The open-system case

By analogy to the closed spin case discussed above one may write the fidelity (56) for the open system (12) as

$$\begin{aligned} F(t) &= \mathrm{Tr}(T_t^2 \circ T_t^1(\rho_+(0))\rho_\psi(t)) \\ &= \sum_{k=0}^{N} \binom{N}{k}\rho_k F_k(t), \end{aligned} \qquad (62)$$

where $F_k(t) := \|U_k(t)\rho_+(0)\|_F^2$. The channels T_t^1 and T_t^2 are specified in Eq. (43) and comments below, respectively. The unitary matrix U_k takes the form $U_k = \exp(-iH_kt)$, with H_k

given by Eq. (45). If we put $\varphi = \pi/2$ as before, then

$$F_k(t) = 1 - \frac{x_k^2}{1 + x_k^2} \sin^2(\Omega(x_k)t), \tag{63}$$

where $x_k := G(N - 2k)/\beta_0 - x$. Using this and Eq. (62) we obtain $F(t) = 1 - R(t)$, where $R(t)$ reads

$$R(t) = \sum_{k=0}^{N} \binom{N}{k} \frac{x_k^2}{1 + x_k^2} \rho_k \sin^2(\Omega(x_k)t). \tag{64}$$

From this equation one can readily see that in the adiabatic domain ($x \to 0$) we have $R(t) \neq 0$ and thus $F(t) < 1$. Therefore, the standard condition that leads to the adiabatic behavior of the closed system (1) does *not* guarantee that in the case of the open-system model (12) it holds true as well. However, if one additionally assumes that coupling with the environment is weak, in comparison with the energy split between the states $|0\rangle$ and $|1\rangle$ of the qubit in question, that is to say, if $G/\beta_0 \ll 1$, then $x_k \ll 1$ (for finite N) and $R(t) \simeq 0$, thus $F(t) \simeq 1$.

VI. SUMMARY

In this article we investigated a qubit in contact with the spin environment and interacting with a rotation magnetic field. The considered model was constructed under a set of assumptions which allow for its exact treatment. We hope that despite the mathematical character the results of the article may serve as starting point for further investigations. The exact models not only can provide a reasonable approximate description of real systems, as is the case for pure dephasing, but also are often used as a basis and inspiration for various improvements [33]. Although the article focuses mainly on mathematical aspects it also includes the example of a natural application of the model in question. This example relates our model to important problems of physics such as, e.g., the problem of the adiabatic approximation for the open quantum systems [34] or the adiabatic quantum computation [35].

We provided the exact reduced dynamics for the system mentioned above. In contrast to the standard methods available in the theory of the open quantum systems our approach was based on the block operator matrices theory. In particular, we resolved the algebraic Riccati equation associated with the Hamiltonian defining the model under consideration. We wish to stress that the method we used in the current article, although applied to the particular model, is general. Nevertheless, its usefulness relies on the ability of solving the Riccati equation. At the present time it is a very difficult task, even for simple systems.

At this point one can ask: How relevant is it to assume a spin environment, instead of a bosonic one, to solve the problem exactly? Is it possible to do so for a bosonic bath? One may pose a more general question: What is a connection between the possibility of obtaining the exact reduced dynamics of the qubit in question and the Hamiltonian specifying the bath? This problem was addressed in Ref. [8]. The results of this article as well as the analysis we carried out suggest that this problem is at least as difficult as resolving the Riccati equation associated with the total Hamiltonian.

Furthermore, we studied the adiabatic approximation for the model in question. It was shown that the standard condition that guarantees the adiabatic evolution in the the case of the closed systems is not valid for the open-system generalization. This is not an unexpected result. It is interesting, however, that the aforementioned condition does ensures the adiabatic behavior of the open system under consideration in the weak-coupling limit.

APPENDIX: PROOF OF EQ. (16)

In order to prove Eq. (16) let us note that the Hamiltonian (12) satisfies the following condition:

$$H(t,\beta) = e^{iKt} H(\beta) e^{-iKt}, \tag{A1}$$

where $K = -\frac{\omega}{2} \sigma_3 \otimes \mathbb{I}_E$. This can be easily proven using the Baker-Campbell-Hausdorff formula [36]. As was shown, in Ref. [2] every quantum system with Hamiltonian $H(t,\beta)$ satisfying (A1) for some Hermitian operator K evolves

$$U_t(\beta) = e^{iKt} e^{-iH_{\text{eff}}(\beta)t}, \quad H_{\text{eff}}(\beta) := H(\beta) + K. \tag{A2}$$

Note that, in general, $[H(\beta), K] \neq 0$ and therefore $[H_{\text{eff}}(\beta), K] \neq 0$. In our case, from Eq. (12) we learn that $H(\beta) = (\beta\sigma_3 + \alpha\sigma_1) \otimes \mathbb{I}_E$, thus

$$\begin{aligned} H_{\text{eff}}(\beta) &= (\beta\sigma_3 + \alpha\sigma_1) \otimes \mathbb{I}_E - \frac{\omega}{2}\sigma_3 \otimes \mathbb{I}_E \\ &= \left[\left(\beta - \frac{\omega}{2}\right)\sigma_3 + \alpha\sigma_1\right] \otimes \mathbb{I}_E \\ &= H\left(\beta - \frac{\omega}{2}\right). \end{aligned} \tag{A3}$$

From Eqs. (A2) and (A3) we have

$$U_t(\beta) = e^{iKt} U_t\left(\beta - \frac{\omega}{2}\right), \tag{A4}$$

where $U_t(\beta)$ is the evolution operator generated by $H(t,\beta)$. Let $\hat{\rho}_t(\beta)$ and $\hat{\eta}_t$ be a density operator for the closed system $Q + E$ associated with the Hamiltonian $H(\beta)$ and $H(t,\beta)$, respectively, in some arbitrary time t. Let us also assume that $\hat{\rho}_0(\beta) = \hat{\eta}_0 \equiv \hat{\rho}$. Using Eq. (A4) one can easily see that

$$\begin{aligned} \hat{\eta}_t &= U_t(\beta)\hat{\rho}U_t^\dagger(\beta) \\ &= e^{iKt} U_t\left(\beta - \frac{\omega}{2}\right)\hat{\rho}U_t^\dagger\left(\beta - \frac{\omega}{2}\right)e^{-iKt} \\ &= \hat{V}_t \hat{\rho}_t\left(\beta - \frac{\omega}{2}\right)\hat{V}_t^\dagger, \end{aligned} \tag{A5}$$

where we introduced $\hat{V}_t = e^{iKt}$. To end the proof we will show that if $\hat{A}_1$, $\hat{A}_2 \in B(\mathcal{H} \oplus \mathcal{H})$ are a 2×2 block operator matrix of the form $\hat{A}_i = A_i \otimes \mathbb{I}_E$, $(i = 1,2)$ and $\hat{B} = [\hat{B}_{ij}] \in B(\mathcal{H} \oplus \mathcal{H})$ then

$$\text{Tr}_E(\hat{A}_1 \hat{B} \hat{A}_2) = A_1 \text{Tr}_E(\hat{B}) A_2. \tag{A6}$$

Equation (A6) follows from the linearity of the trace $\text{Tr}(\cdot)$ operation and the definition (6) of partial trace. Note that $\hat{V}_t = V_t \otimes \mathbb{I}_E$, where V_t is given by Eq, (17), thus taking a partial trace of Eq. (A5) and using (A6) we obtain (16) with V_t given by (17).

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