Sub- and super-fidelity as bounds for quantum fidelity

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Quantum states and fidelity

Quantum state is an operator $\rho : \mathcal{H}_N \to \mathcal{H}_N$, which is positive semi-definite ($\rho \geq 0$) and normalised ($\mathrm{tr}\rho = 1$). We denote by $\Omega_N \subset \mathcal{M}_N$ the space of density matrices on $\mathbb{C}^N$.

We define the fidelity between two states as

$$F(\rho_1, \rho_2) = (\mathrm{tr}|\sqrt{\rho_1}\sqrt{\rho_2}|)^2 = ||\rho_1^{1/2}\rho_2^{1/2}||_1^2,$$

where $||\cdot||_1$ is a trace norm, *i.e.* $||A||_1 = \mathrm{tr}|A| = \sum_{i=1}^{N} \sigma_i(A)$.

In the case of two pure states $\rho_1 = |\phi\rangle\langle\phi|, \rho_2 = |\psi\rangle\langle\psi|$ we have $F(\rho_1, \rho_2) = |\langle\psi|\phi\rangle|^2$.

Properties of fidelity

Fidelity has few nice properties

- **Bounds:** $0 \leq F(\rho_1, \rho_2) \leq 1$. Furthermore $F(\rho_1, \rho_2) = 1$ iff $\rho_1 = \rho_2$, while $F(\rho_1, \rho_2) = 0$ iff $\mathrm{supp}(\rho_1) \perp \mathrm{supp}(\rho_2)$.
- **Symmetry:** $F(\rho_1, \rho_2) = F(\rho_2, \rho_1)$.
- **Unitary invariance:** $F(\rho_1, \rho_2) = F(U\rho_1 U^\dagger, U\rho_2 U^\dagger)$, for any unitary operator U.
- **Concavity:** $F(\rho, a\rho_1 + (1-a)\rho_2) \geq aF(\rho, \rho_1) + (1-a)F(\rho, \rho_2)$, for $a \in [0, 1]$.
- **Multiplicativity:** $F(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) = F(\rho_1, \rho_3)F(\rho_2, \rho_4)$.
- **Joint concavity:** $\sqrt{F}(a\rho_1 + (1-a)\rho_2, a\rho_1' + (1-a)\rho_2') \geq a\sqrt{F}(\rho_1, \rho_1') + (1-a)\sqrt{F}(\rho_2, \rho_2')$, for $a \in [0, 1]$.

Classical counterpart

Fidelity between two diagonal operators is equal to the *Bhattacharyya* coefficient B for their eigenvalues.

$$\sqrt{F}(\mathrm{diag}(\rho_1), \mathrm{diag}(\rho_2)) = B(p, q) = \sum_{i=1}^{n} \sqrt{p_i q_i}$$

Here p and q are eigenvalues of ρ_1 and ρ_2 respectively.

Fidelity as function of eigenvalues

We start our analysis of fidelity by expressing it in terms of eigenvalues $\lambda_i, \; i = 1, \ldots, N$ of the (positive) matrix $\sqrt{\rho_1^{1/2}\rho_2\rho_1^{1/2}}$. Using the fact that matrix $\rho_1\rho_2$ is similar to matrix $\sqrt{\rho_1}\rho_2\sqrt{\rho_2}$ one can write

$$\sqrt{F(\rho_1,\rho_2)} = \mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} = \sum_{i=1}^{N}\lambda_i,$$

and since $\mathrm{tr}\rho_1\rho_2 = \mathrm{tr}\sqrt{\rho_1}\rho_2\sqrt{\rho_1} = \sum_{i=1}^{N}\lambda_i^2$ by squaring the above we get

$$F(\rho_1,\rho_2) = \left(\sum_{i=1}^{N}\lambda_i\right)^2 = \mathrm{tr}\rho_1\rho_2 + 2\sum_{i<j}\lambda_i\lambda_j.$$

Elementary symmetric functions

For a give matrix $X \in \mathcal{M}_N$ with eigenvalues $\lambda_1, \ldots, \lambda_N$ we define elementary symmetric function $s_m(X)$ as $s_m(\lambda_1, \ldots, \lambda_N)$
For example

$$s_2(X) = \sum_{i<j} \lambda_i \lambda_j,$$
$$s_3(X) = \sum_{i<j<k} \lambda_i \lambda_j \lambda_k.$$

Using this notion we can write the fidelity as

$$F(\rho_1, \rho_2) = \mathrm{tr} \rho_1 \rho_2 + 2 s_2\left(\sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}}\right).$$

Lower bound by Uhlmann

In his unpublished work Uhlmann suggested an inequality

$$F(\rho_1,\rho_2) \geq \mathrm{tr}\rho_1\rho_2 + \sqrt{2}\sqrt{(\mathrm{tr}\rho_1\rho_2)^2 - \mathrm{tr}\rho_1\rho_2\rho_1\rho_2}.$$

We define sub-fdelity as

$$E(\rho_1,\rho_2) = \rho_1\rho_2 + \sqrt{2}\sqrt{(\mathrm{tr}\rho_1\rho_2)^2 - \mathrm{tr}\rho_1\rho_2\rho_1\rho_2}.$$

Using elementary symmetric functions this quantity can be represented as

$$E(\rho_1,\rho_2) = \mathrm{tr}\rho_1\rho_2 + 2\sqrt{s_2(\rho_1\rho_2)}.$$

Super-fdelity

We can introduce upper bound which is complementary to sub-fidelity[1]

$$F(\rho_1, \rho_2) \leq \mathrm{tr}\rho_1\rho_2 + \sqrt{(1 - \mathrm{tr}\rho_1^2)(1 - \mathrm{tr}\rho_2^2)}.$$

Again we can use elementary symetric function to get compact expression for super-fidelity

$$G(\rho_1, \rho_2) = \mathrm{tr}\rho_1\rho_2 + \sqrt{(1 - \mathrm{tr}\rho_1^2)(1 - \mathrm{tr}\rho_2^2)} = \mathrm{tr}\rho_1\rho_2 + 2\sqrt{s_2(\rho_1)s_2(\rho_2)}.$$

thus we have

$$s_2(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}) \leq \sqrt{s_2(\rho_1)s_2(\rho_2)}$$

[1] J. A. M, Z. Puchała, P. Horodecki, A. Uhlmann, K. Życzkowski, QI&C, **9**, 1&2 (2009), arXiv:0805.2037.

Two inequalities $E(\rho_1, \rho_2) \leq F(\rho_1, \rho_2) \leq G(\rho_1, \rho_2)$ can be written in a compact way using elementay symetric functions

$$\sqrt{s_2(\rho_1\rho_2)} \leq s_2\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right) \leq \sqrt{s_2(\rho_1)s_2(\rho_2)}$$

- If of the states is pure we have equality $E = F = G$.
- Moreover these quantities coincide for one-qubit states $(N = 2)$.

Properties of sub- and super-fidelity

Sub- and super-fidelity share some properties with fidelity

- **i')** **Bounds:** $0 \le E(\rho_1, \rho_2) \le 1$ oraz $0 \le G(\rho_1, \rho_2) \le 1$.
- **ii')** **Symmetry:** $E(\rho_1, \rho_2) = E(\rho_2, \rho_1)$ and $G(\rho_1, \rho_2) = G(\rho_2, \rho_1)$.
- **iii')** **Unitary invariance:** $E(\rho_1, \rho_2) = E(U\rho_1 U^\dagger, U\rho_2 U^\dagger)$ and $G(\rho_1, \rho_2) = G(U\rho_1 U^\dagger, U\rho_2 U^\dagger)$, for any unitary U.
- **iv')** **Concavity:** Sub– and super–fidelity are concave,

$$
\begin{aligned}
E(A, \alpha B + (1-\alpha)C) &\ge \alpha E(A, B) + (1-\alpha)E(A, C), \\
G(A, \alpha B + (1-\alpha)C) &\ge \alpha G(A, B) + (1-\alpha)G(A, C).
\end{aligned}
$$

- **v')** Super-fidelity (just like $\sqrt{F}$) is **jointly concave** in its two arguments

$$\sqrt{F}(a\rho_1 + (1-a)\rho_2, a\rho_1' + (1-a)\rho_2') \ge a\sqrt{F}(\rho_1, \rho_1') + (1-a)\sqrt{F}(\rho_2, \rho_2')$$

for $a \in [0, 1]$.

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vii') Sub-fidelity is sub-multiplicative

$$E(\rho_1 \otimes \rho_2, \rho_3 \otimes \rho_4) \leq E(\rho_1, \rho_3)E(\rho_2, \rho_4).$$

Classical case

If the density matrices ρ_p and ρ_q commute, discussed bound can be expressed in terms of respective eigenvalues $\{p_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^N$:

$$
\begin{aligned}
E(\rho_p, \rho_q) &= \sum_{i=1}^N p_i q_i + \sqrt{2\left[\left(\sum_{i=1}^N p_i q_i\right)^2 - \sum_{i=1}^N p_i^2 q_i^2\right]}, \\
F(\rho_p, \rho_q) &= \left(\sum_{i=1}^N \sqrt{p_i q_i}\right)^2, \\
G(\rho_p, \rho_q) &= \sum_{i=1}^N p_i q_i + \sqrt{\left(1 - \sum_{i=1}^N p_i^2\right)\left(1 - \sum_{i=1}^N q_i^2\right)}.
\end{aligned}
$$

Mixed states

To get some feeling about behaviour of E and G we calculate them for states of the form

$$\rho_a = a|\psi\rangle\langle\psi| + (1-a)I/N.$$

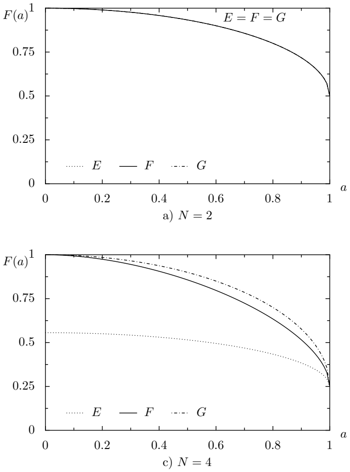
where $|\psi\rangle$ is an arbitrary pure state.
For $\rho_* := I/N$ we get

$$F(\rho_a, \rho_*) = \frac{1}{N^2}\left(\sqrt{(N-1)a+1} + (N-1)\sqrt{1-a}\right)^2,$$

and sub- and super-fidelity are expressed as

$$\begin{aligned} E(\rho_a, \rho_*) &= \frac{1}{N} + \sqrt{2}\frac{1}{N}\sqrt{1-\frac{1}{N}}\sqrt{1-a^2}, \\ G(\rho_a, \rho_*) &= \frac{1}{N} + \left(1-\frac{1}{N}\right)\sqrt{1-a^2}. \end{aligned}$$

Comparison of sub–fidelity E, fidelity F and super–fidelity G.



Difference $G - F$ and $E - F$

F and G coincide if one of the states is pure, but it is natural to ask how big the difference $G - F$ might be.
Let us use the Hilbert space of dimension $N = 2M$ and states $\rho_1 = \frac{2}{N}\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ and $\rho_2 = \frac{2}{N}\mathrm{diag}(0, \ldots, 0, 1, \ldots, 1)$.

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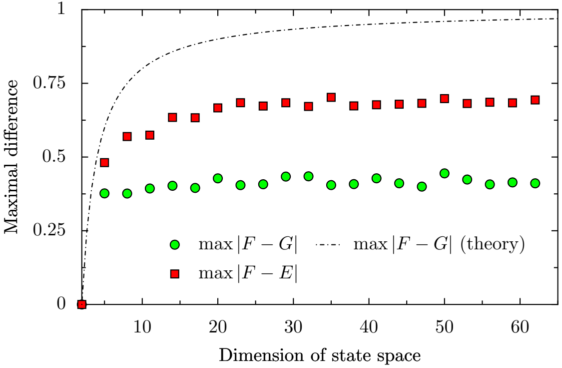
Since they are supported by orthogonal subspaces their fidelity vanishes, $F(\rho_1, \rho_2) = 0$. On the other hand their super–fidelity is equal to

$$G(\rho_1, \rho_2) = \frac{N-2}{N},$$

and the difference $F - G$ can be arbitrarily close to 1.

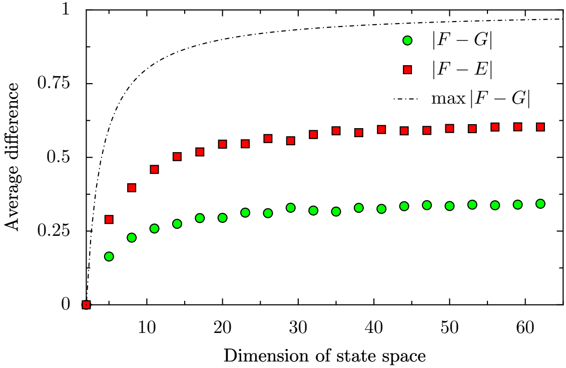
Maximal difference



Max difference between fidelity and sub- and super- fidelity for random states in dimesnions $N = 2, 3, \dots, 62$.

Average difference

On average situation looks somehow better.



Average difference between fidelity and sub- and super- fidelity for some values of $N \in [2, 62]$.

Super-fidelity and trace distance

For any $\rho_1, \rho_2 \in \Omega_N$ super-fidelity and trace distance are related by the inequality[2]

$$1 - G(\rho_1, \rho_2) \leq D_{\mathrm{tr}}(\rho_1, \rho_2)$$

Probability of error for distinguishing two density matrices $\rho_1, \rho_2 \in \Omega_N$ is expressed by the trace distance as

$$P_E(\rho_1, \rho_2) = \frac{1}{2}(1 - D_{\mathrm{tr}}(\rho_1, \rho_2)).$$

Using the above inequalities we can write

$$\frac{1}{2} G(\rho_1, \rho_2) \geq P_E(\rho_1, \rho_2)$$

[2] Z. Puchała, J. A. M., *Bounds on trace distance based on super-fidelity*, in preparation

Experimental setup for measuring super-fidelity

We use fact that $\mathrm{tr} V_{12}\rho_1 \otimes \rho_2 = \mathrm{tr}\rho_1\rho_2$ where V_{12} is a SWAP operator. $V_{12} = P_{12}^{+} - P_{12}^{-}$ is hermitian and thus represents an observable.

To measure G we need a source which creates pairs $\rho_i \otimes \rho_j$, $i, j = 1, 2$.

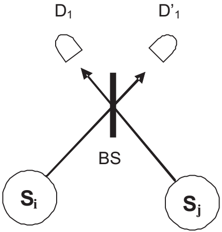
The probability of measuring P_{12}^{-} reads $p_{ij}^{-} = \mathrm{tr} P_{12}^{-}\rho_i \otimes \rho_j$ and using it we can write

$$G = 1 - 2(p_{12}^{-} - \sqrt{p_{11}^{-} - p_{22}^{-}})$$

.

Experimental setup for measuring super-fidelity

Probability of the event that both detectors click is equal to p_{ij}^{-}. On detectors clicks with $p_{ij}^{+} = 1 - p_{ij}^{-}$. Beam-splitter projects on P^{-} or P^{+}

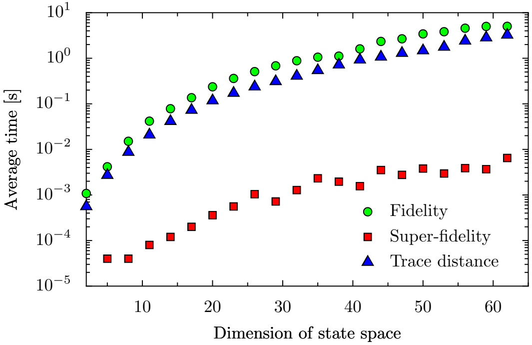


The experimental setup is in this case very simple.[3]

[3] F. A. Bovino et all, PRL **95**, 240407 (2005)

Computational efficiency

E and G are much easier to calculate than fidelity F. To compute any of these bounds it is enough to evaluate three traces only.



(See also P. E. M. F. Mendonca, *et all*, arXiv:0806.1150)

Conclusions

- Proposed bounds share with fidelity its main features (they are bounded, symmetric, unitary invariant and concave).
- Super–fidelity G can be used in place of fidelity F for small systems or when at least one of the states is pure enough.
- Sub- and super-fidelity can be (in principle) measured in laboratory.
- It is easy to calculate them using standard computer algebra systems.

References

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Thank you.