Product (aka local) numerical range and its applications in quantum information theory

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Overview

Motivation

Standard definitions

Product version of the numerical range

Applications in quantum information
- Local distinguishability of unitary operators
- Calculation of minimum output entropy
- Identification of positive maps

Summary

Motivation

Unitary orbits vs local unitary orbits

Global case

Maximize fidelity $F(A,B) = \mathrm{tr}|\sqrt{A}\sqrt{B}|$ using unitary operations

$$\max_{U} F(UAU^{\dagger}, B)$$

Local case

Maximize fidelity between bipartite states using unitary operations of the form $U = U_A \otimes U_B$

$$\max_{U \in \mathbf{SU}(m) \otimes \mathbf{SU}(n)} F(UAU^{\dagger}, B)$$

If one of the states is product and pure (rank 1 projector) we have

$$\max_{U \in \mathbf{SU}(m) \otimes \mathbf{SU}(n)} F(U|\psi\rangle\langle\psi|U^\dagger, B) = \max_{|\psi\rangle, |\phi\rangle} \langle\psi \otimes \phi|B|\psi \otimes \phi\rangle.$$

This answers the question about the best approximation of B using bipartite pure states. But...

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- The definition can be extended to any matrix B – not necessarily positive, Hermitian...
- We can also introduce other decomposition of the underlying Hilbert space...

Some standard definitions

Let A be a matrix acting on complex vector space $\mathbb{C}^n$. We define *numerical range* (or field of values) of A as

$$\mathbf{\Lambda}(A) = \{\langle v|A|v\rangle : v \in \mathbb{C}^n, ||v|| = 1\}.$$

$\mathbf{\Lambda}(A)$ is a convex, compact subset of a complex plane. *Numerical radius* defined as

$$\mathbf{r}(A) = \max\{|z| \in \mathbf{\Lambda}(A)\}$$

is a norm on the space of operators. For normal operators B we have

$$\mathbf{\Lambda}(B) = \mathrm{Co}(\sigma(B)),$$

(this is iff) and for Hermitian operator C we have

$$\mathbf{\Lambda}(C) = [\lambda_{\mathrm{min}}(C), \lambda_{\mathrm{max}}(C)].$$

Definition (Product numerical range)

Let $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$ be a tensor product Hilbert space. We define the *product numerical range* $\mathbf{\Lambda}^{\otimes}(\mathrm{X})$ of X, with respect to this tensor product structure, as

$$\mathbf{\Lambda}^{\otimes}(X) = \{\langle \psi_A \otimes \psi_B | X | \psi_A \otimes \psi_B \rangle : |\psi_A\rangle \in \mathcal{H}_K, |\psi_B\rangle \in \mathcal{H}_M\},$$

where $|\psi_A\rangle \in \mathcal{H}_K$ and $|\psi_B\rangle \in \mathcal{H}_M$ are normalized.

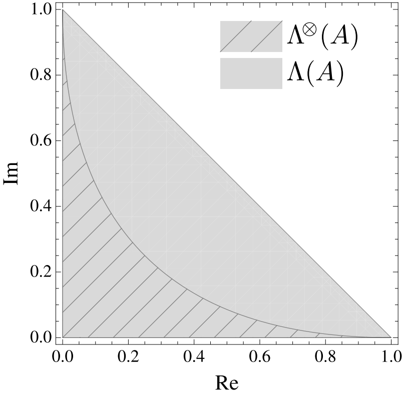
Note

This definition depends on decomposition. In most cases we will stick to bipartite systems, but in multipartite systems everything is even more interesting.

- $\mathbf{\Lambda}^{\otimes}(A)$ is compact – it is a continuous image of a connected set
- $\mathbf{\Lambda}^{\otimes}(A)$ is **not necessarily convex** (an example on the next slide)
- $\mathbf{\Lambda}^{\otimes}(A)$ is **not necessarily simply connected** (for multipartite systems)

Consider normal matrix with eigenvalues $0, 1$ and i

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$



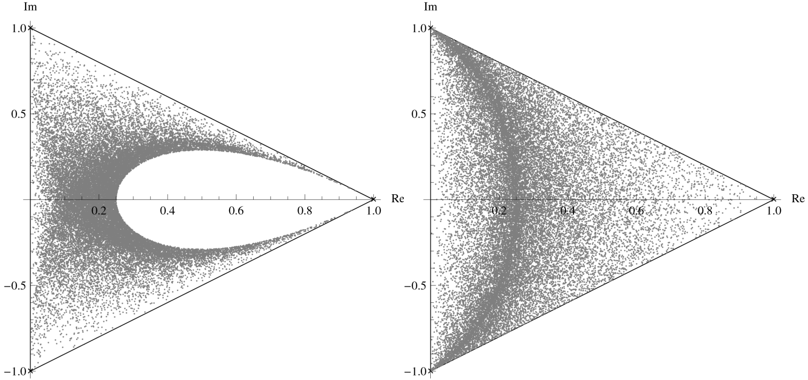
In this case $\mathbf{\Lambda}^{\otimes}(A) = \left\{ x + yi : 0 \leq x, 0 \leq y, \sqrt{x} + \sqrt{y} \leq 1 \right\}$

- In the case of a tripartite system, product numerical range is not necessarily simply connected.
- For a four-partite system $\mathbf{\Lambda}^{\otimes}(A)$ can have genus 2.

Question

How does this behavior scale with the number of subsystems?

Let us consider two matrices A_1 and A_2 defined as
$A_1 = \mathrm{diag}(1, i, i, -\mathbf{i}, \mathbf{i}, -i, -i, 1)$ and $A_2 = \mathrm{diag}(1, i, i, \mathbf{i}, -\mathbf{i}, -i, -i, 1)$.

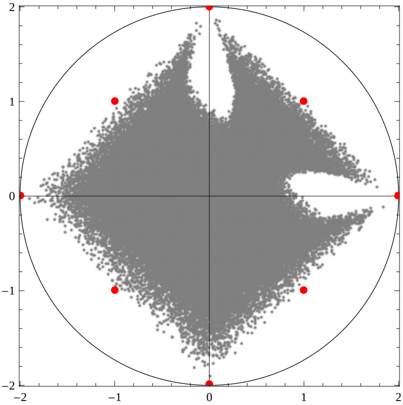


Local numerical range of A_1 and A_2. Numerical range is bounded by solid lines.

In the 4-partite system $\mathbf{\Lambda}^{\otimes}(A)$ can have genus 2 *i.e.* it can be a connected sum of two tori. Let us consider the matrix defined as

$$\mathrm{diag}(2, 1+i, 1+i, 2i, -1+i, -2, -2i, 1-i, -1+i, -2i, -2, 1-i, 2i, -1-i, -1-i, 2).$$

with eigenvalues $\{-2, -1-i, -1+i, -2i, 2i, 1-i, 1+i, 2\}$



But in most cases we need to deal with Hermitian (or positive defined) matrices...
Let A be Hermitian acting on $\mathcal{H}_M \otimes \mathcal{H}_N$.

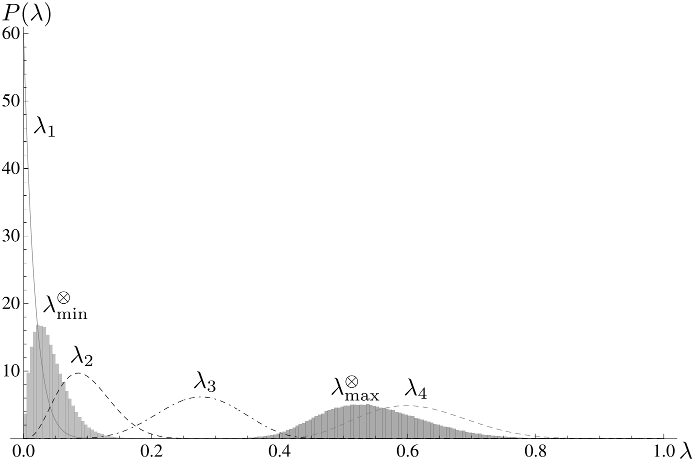
- $\mathbf{\Lambda}^{\otimes}(A) \subset \mathbf{\Lambda}(A)$ and is reduced to one point iff $A = \alpha \mathbb{1}$
- $\mathbf{\Lambda}^{\otimes}(A) = [\lambda^{\otimes}_{\min}, \lambda^{\otimes}_{\max}]$ – *i.e.* it is a line segment

More precise localization of $\mathbf{\Lambda}^{\otimes}(A)$ for $\mathcal{H}_M \otimes \mathcal{H}_K$.

$\lambda^{\otimes}_{\max} \geq \lambda_{K+M-1}$ and $\lambda^{\otimes}_{\min} \leq \lambda_{(K-1)(M-1)+1}$

Example

Probability density of eigenvalues and product values $\lambda^{\otimes}_{\min}, \lambda^{\otimes}_{\max}$ for random two-qubit density matrix, generated according to the Hilbert-Schmidt measure.



2×2 Example

For a family of matrices $X_{t,s}$ with $s, t \geq 0$

$$X_{t,s} = \begin{bmatrix} 2 & 0 & 0 & t \\ 0 & 1 & s & 0 \\ 0 & s & -1 & 0 \\ t & 0 & 0 & -2 \end{bmatrix}$$

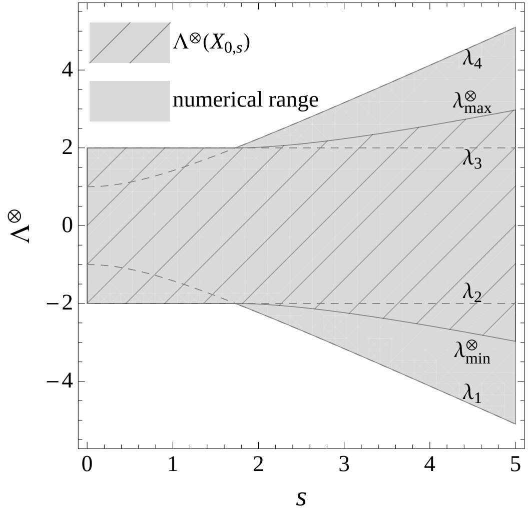
we obtained the exact values of $\lambda_{\min}^{\otimes}$ and $\lambda_{\max}^{\otimes}$. The exact formula for local numerical range reads

$$\mathbf{\Lambda}^{\otimes}(X_{t,s}) = [-a(t+s), a(t+s)],$$

where

$$a(t) = \begin{cases} 2 & \text{for} \quad t \in \left[0, \sqrt{3}\right) \\ \frac{\sqrt{t^4+10t^2+9}}{2t} & \text{for} \quad t \in \left[\sqrt{3}, \infty\right) \end{cases}.$$

Numerical range and local numerical range for matrices $X_{0,s}$.



Applications in quantum information

Exemplary applications

- local distinguishability of unitary operators
- minimum output entropy
- identification of positive maps

Problem

Distinguish two unitary operators U and V using product states.

This is to say: find a product state $|\phi \otimes \psi\rangle$ such that $U|\phi \otimes \psi\rangle$ is orthogonal to $V|\phi \otimes \psi\rangle$

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Connection with product range

Operators U (on $\mathcal{H}_M$) and V (on $\mathcal{H}_N$) are distinguishable iff $0 \in \mathbf{\Lambda}^{\otimes}(U^\dagger V)$. If this is the case we have

$$\langle \phi \otimes \psi | U^\dagger V | \phi \otimes \psi \rangle = 0$$

for some $|\psi\rangle \in \mathcal{H}_M$ and $|\phi\rangle \in \mathcal{H}_N$.

Problem

Optimize minimal output entropy of quantum channel.

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Connection with product range

- **1-qubit maps:** If $\lambda^{\otimes}_{\min}$ is the smallest local eigenvalue of D_Φ then

$$S_{\min}(\Phi) = -\lambda^{\otimes}_{\min} \log \lambda^{\otimes}_{\min} - (1-\lambda^{\otimes}_{\min}) \log(1-\lambda^{\otimes}_{\min})$$

- **general case:** If $1 \in \mathbf{\Lambda}^{\otimes}(D_\Phi)$ then $S_{\min}(\Phi) = 0$

Question

Is it possible to obtain non-additivity for 1-qubit channels?

Example

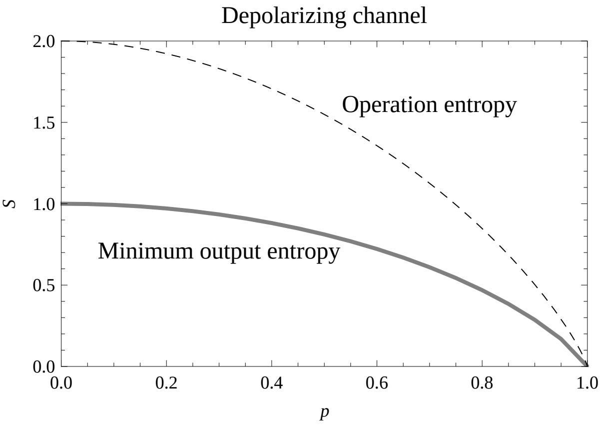
For depolarizing channel we have the following dynamical matrix

$$D_\Phi = \begin{pmatrix} \frac{p+1}{2} & 0 & 0 & p \\ 0 & \frac{1-p}{2} & 0 & 0 \\ 0 & 0 & \frac{1-p}{2} & 0 \\ p & 0 & 0 & \frac{p+1}{2} \end{pmatrix},$$

for $p \in [0,1]$. In this case we have $\lambda^{\otimes}_{\min}(D_\Phi) = \frac{1}{2}(1-p)$ and thus

$$S_{\min}(\Phi) = -\frac{\log\left(\frac{1}{4} - \frac{p^2}{4}\right) + 2p \tanh^{-1}(p)}{\log(4)}.$$

MOE vs operation entropy



Minimum output entropy for one-qubit depolarizing channel.

Problem
Entanglement can be detected using positive but not completely positive maps.

Peres-Horodecki criterion
Bipartite state ρ is separable iff $(\Phi \otimes \mathbb{1})\rho \geq 0$ for all positive Φ.

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Connection with product range

Map Φ is positive iff $\mathbf{\Lambda}^{\otimes}(D_{\Phi}) \subset [0, \infty)$.

Summary

- Product numerical range can be used to study various problems in quantum information theory (see also: T. Schulte-Herbrüggen *et al.*, Lin. Multilin. Alg. 56 (2008) 3).
- In some cases we are able to describe $\mathbf{\Lambda}^{\otimes}(A)$ analytically, but usually we have to rely on numerical results.
- Generalization: separable numerical range (Duan *et al.* PRL 100, 020503 (2008)). In general: product states are one one of possible choices...
- Still many open problems, including topological properties (genus for multipartite systems).

Thank you for your attention!

- For physicists - arXiv:0905.3646
- For mathematicians - arXiv:1008.3482
- Some related functions in Mathematica
  `http://bit.ly/qi-mathematica`