Bounds on fidelity and distinguishability of quantum states

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State discrimination
- Measures
- Distinguishability

Distingushability measures
- Bures distance
- Trace distance

Results: Bounds on measures
- Bures distance
- Trace distance

Summary

State discrimination

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Problem: One can use different distance measures to distinguish states (and probability distributions).

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- trace distance $D_{\mathrm{tr}}(A, B) = \frac{1}{2}\mathrm{tr}|A-B|$ (Shatten 1-norm)
- Bures distance $D_B(A, B) = \sqrt{2(1-\mathrm{tr}|\sqrt{A}\sqrt{B}|)}$, which is the function of fidelity $F(A, B) = \left(\mathrm{tr}\left[\sqrt{\sqrt{A}B\sqrt{A}}\right]\right)^2$

State discrimination

- $\mathrm{supp}(A) \perp \mathrm{supp}(B) \Leftrightarrow D_{\mathrm{tr}}(A, B) = 1 \Leftrightarrow D_B(A, B) = \sqrt{2}$
- for $N > 2$ states can be arbitrary close wrt D_{HS} and even then have orthogonal supports $\Rightarrow$ Hilbert-Schmidt Distance does not provide good measure of distinguishability

State discrimination

Perfect state discrimination

It is impossible to discriminate among two pure state if they are not orthogonal!

Mixed states A and B can be deterministically discriminated iff their supports do not overlap.

Distinguishability

- Distinguishability of states $\equiv$ trace distance between states[1]
- $1-\sqrt{F(A,B)} \leq D_{\mathrm{tr}}(A,B) \leq \sqrt{1-F(A,B)}$ and thus $F(A,B)=0 \Rightarrow$ states A and B can be perfectly distinguished

[1] B.-G. Englert, PRL **96**, 040501 (1996).

Distinguishability

Geometrical consequences

Let R be an arbitrary convex subset of $\mathcal{M}_N$. Assume that there exists a simplex $\Delta_k \subset R$ and assume that R does not contain Δ_{k+1}.
Then the maximal number of states of R which can be discriminated deterministically is equal to k.

Distinguishability

Geometrical picture

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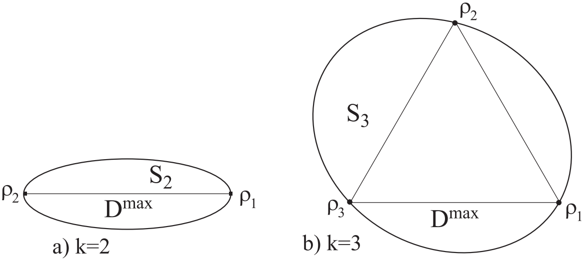

Measures

Different measures generate different geometry

- $\mathbb{C}^2$ (qubit) + Hilbert-Schmidt (or trace) distance = Bloch sphere
- for even N in N dimensional space Hilbert-Schmidt distance between states $diag(0, \ldots, 0, N/2, \ldots, N/2)$ and $diag(N/2, \ldots, N/2, 0, \ldots, 0)$ reads $2/\sqrt{N}$ – but those states have orthogonal supports
- the diameter of the state set – maximal distance between states

$$D_B^{max} = \sqrt{2}, \;\; D_{\mathrm{tr}}^{max} = 1, \;\; D_{HS}^{max} = \sqrt{2}$$

Bures distance

The classical analogue of Bures distance reads

$$D_B^C(p,q) := \sqrt{2(1 - B(p,q))}$$

where

$$B(p,q) = \sum_{i=1}^{n} \sqrt{p_i q_i}$$

is the *Bhattacharayya* coefficient.

Trace distance

Trace distance is equal to L_1 distance between probability vectors.

$$D_{\mathrm{tr}}^{C}(p, q) := \sum_{i=1}^{n} |p_i - q_i|$$

Distance between orbits

Proposition: Maximum and minimum are obtained for classical states

$$M(A,B) := \max_{U,V}(UAU^\dagger, VBV^\dagger) = D(p^\uparrow, q^\downarrow)$$

$$m(A,B) := \min_{U,V}(UAU^\dagger, VBV^\dagger) = D(p^\uparrow, q^\uparrow)$$

where $p^\uparrow$ and $q^\downarrow$ are ordered eigenvalues of A and B respectively.

Bounds for Fidelity

Upper bound

It can be shown that

$$(p^{\uparrow})^s(q^{\downarrow})^r \leq \mathrm{tr} A^s B^r \leq (p^{\uparrow})^s(q^{\uparrow})^r$$

Since $\mathrm{tr}|\sqrt{A}\sqrt{B}| \geq \mathrm{tr}\sqrt{A}\sqrt{B}$ we have

$$F(A,B) \leq \sqrt{p^{\uparrow}}\sqrt{q^{\uparrow}}$$

Bounds for Fidelity

Lower bound

We use von Neumann inequality [2]

$$|\mathrm{tr} AB| \leq \sum_{i=1}^{n} \sigma_i(A)\sigma_i(B)$$

and equation $\mathrm{tr}|A| = |\max_U(UA)|$

[2] L. Mirsky, *Monatshefte für Mathematik*, **79**, 303 (1973).

Bounds for trace distance

Upper bound

To prove this bound we use (again) equation $\mathrm{tr}|A| = \max_U |\mathrm{tr}\, UA|$, which is in fact true for any partial isometry (rank $k < n$ projection operator).

$$\mathrm{tr}|A - B| = \max_U \left| \sum_{i=1}^{n} p_i \langle \mu_i | U | \mu_i \rangle - q_i \langle \nu_i | U | \nu_i \rangle \right|$$

Bounds for trace distance

Lower bound

This bound is the simple conclusion from the theorem[3] about singular values of the operators difference

$$\sum_{i=1}^{n} |\sigma_i(A) - \sigma_i(B)| \leq \sum_{i=1}^{n} \sigma_i(A - B),$$

where σ_i are ordered in nondecreasing order.

[3] Horn, Johnson, *Topics in Matrix Analysis*, Cambridge, 1991

Weyl chamber

Maximal distance is achieved for points belonging to the opposite chambers

The minimal distance m between the uniray orbits of two quantum states are equal to the distances between the corresponding spectra a and b belonging to the same Weyl chamber shown for a) $N = 2$ and b) $N = 3$.

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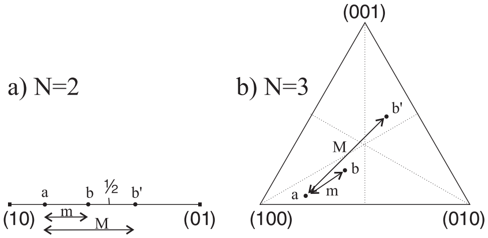

Summary

- $\sum_{i=1}^{n} \sqrt{p^{\uparrow} q^{\downarrow}} \leq F(A,B) \leq \sum_{i=1}^{n} \sqrt{p^{\uparrow} q^{\uparrow}}$ which gives

$$\sum_{i=1}^{n} \sqrt{p_i^{\uparrow} q_i^{\uparrow}} \leq D_B(A,B) \leq \sum_{i=1}^{n} \sqrt{p_i^{\uparrow} q_i^{\downarrow}}$$

  for Bures distance
- for trace distance we have

$$\sum_{i=1}^{n} |p_i^{\uparrow} - q_i^{\uparrow}| \leq \mathrm{tr}|A-B| \leq \sum_{i=1}^{n} |p_i^{\uparrow} - q_i^{\downarrow}|$$

More details in arXiv:0711.4286

Final remarks

- Hilber-Schmidt distance does not provide good measure of distingushability
- Bures distance and trace distance can determinate when the states can be deterministically discriminated
- looking for distinguishable states in a rotationally invariant subset it is sufficient to analyze the set of "classical" density matrices

Thank you!