

Bounds on fidelity and distinguishability of quantum states

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State discrimination

Measures

Distinguishability

Distinguishability measures

Bures distance

Trace distance

Results: Bounds on measures

Bures distance

Trace distance

Summary

State discrimination

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Problem: One can use different distance measures to distinguish states (and probability distributions).

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- ▶ trace distance $D_{\text{tr}}(A, B) = \frac{1}{2}\text{tr}|A - B|$ (Shatten 1-norm)
- ▶ Bures distance $D_B(A, B) = \sqrt{2(1 - \text{tr}|\sqrt{A}\sqrt{B}|)}$, which is the function of fidelity $F(A, B) = \left(\text{tr} \left[\sqrt{\sqrt{A}B\sqrt{A}} \right]\right)^2$

State discrimination

- ▶ $\text{supp}(A) \perp \text{supp}(B) \Leftrightarrow D_{\text{tr}}(A, B) = 1 \Leftrightarrow D_B(A, B) = \sqrt{2}$
- ▶ for $N > 2$ states can be arbitrary close wrt D_{HS} and even then have orthogonal supports \Rightarrow Hilbert-Schmidt Distance does not provide good measure of distinguishability

State discrimination

Perfect state discrimination

It is impossible to discriminate among two pure state if they are not orthogonal!

Mixed states A and B can be deterministically discriminated iff their supports do not overlap.

Distinguishability

- ▶ Distinguishability of states \equiv trace distance between states¹
- ▶ $1 - \sqrt{F(A, B)} \leq D_{\text{tr}}(A, B) \leq \sqrt{1 - F(A, B)}$ and thus $F(A, B) = 0 \Rightarrow$ states A and B can be perfectly distinguished

¹B.-G. Englert, PRL **96**, 040501 (1996).

Distinguishability

Geometrical consequences

Let R be an arbitrary convex subset of \mathcal{M}_N . Assume that there exists a simplex $\Delta_k \subset R$ and assume that R does not contain Δ_{k+1} .

Then the maximal number of states of R which can be discriminated deterministically is equal to k .

Distinguishability

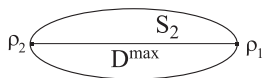
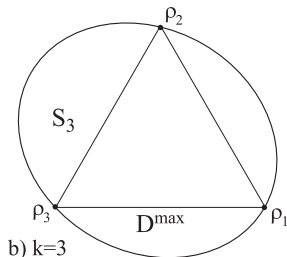
Geometrical picture

Distinguishable states which form a maximal simplex of size k with side length D^{\max} , with respect to the Bures (or the trace) metric.

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a) $k=2$ b) $k=3$

Measures

Different measures generate different geometry

- ▶ \mathbb{C}^2 (qubit) + Hilbert-Schmidt (or trace) distance = Bloch sphere
- ▶ for even N in N dimensional space Hilbert-Schmidt distance between states $\text{diag}(0, \dots, 0, N/2, \dots, N/2)$ and $\text{diag}(N/2, \dots, N/2, 0, \dots, 0)$ reads $2/\sqrt{N}$ – but those states have orthogonal supports
- ▶ the diameter of the state set – maximal distance between states

$$D_B^{\max} = \sqrt{2}, \quad D_{\text{tr}}^{\max} = 1, \quad D_{\text{HS}}^{\max} = \sqrt{2}$$

Bures distance

The classical analogue of Bures distance reads

$$D_B^C(p, q) := \sqrt{2(1 - B(p, q))}$$

where

$$B(p, q) = \sum_{i=1}^n \sqrt{p_i q_i}$$

is the *Bhattacharaya* coefficient.

Trace distance

Trace distance is equal to L_1 distance between probability vectors.

$$D_{\text{tr}}^{\mathcal{C}}(p, q) := \sum_{i=1}^n |p_i - q_i|$$

Distance between orbits

Proposition: Maximum and minimum are obtained for classical states

$$M(A, B) := \max_{U, V} (UAU^\dagger, VBV^\dagger) = D(p^\uparrow, q^\downarrow)$$

$$m(A, B) := \min_{U, V} (UAU^\dagger, VBV^\dagger) = D(p^\downarrow, q^\uparrow)$$

where p^\uparrow and q^\downarrow are ordered eigenvalues of A and B respectively.

Bounds for Fidelity

Upper bound

It can be shown that

$$(p^\uparrow)^s (q^\downarrow)^r \leq \text{tr} A^s B^r \leq (p^\uparrow)^s (q^\uparrow)^r$$

Since $\text{tr} |\sqrt{A}\sqrt{B}| \geq \text{tr} \sqrt{A}\sqrt{B}$ we have

$$F(A, B) \leq \sqrt{p^\uparrow} \sqrt{q^\uparrow}$$

Bounds for Fidelity

Lower bound

We use von Neumann inequality²

$$|\operatorname{tr}AB| \leq \sum_{i=1}^n \sigma_i(A)\sigma_i(B)$$

and equation $\operatorname{tr}|A| = |\max_U \operatorname{tr}(UA)|$

²L. Mirsky, *Monatshefte für Mathematik*, **79**, 303 (1973).

Bounds for trace distance

Upper bound

To prove this bound we use (again) equation $\text{tr}|A| = \max_U |\text{tr}UA|$, which is in fact true for any partial isometry (rank $k < n$ projection operator).

$$\text{tr}|A - B| = \max_U \left| \sum_{i=1}^n p_i \langle \mu_i | U | \mu_i \rangle - q_i \langle \nu_i | U | \nu_i \rangle \right|$$

Bounds for trace distance

Lower bound

This bound is the simple conclusion from the theorem³ about singular values of the operators difference

$$\sum_{i=1}^n |\sigma_i(A) - \sigma_i(B)| \leq \sum_{i=1}^n \sigma_i(A - B),$$

where σ_i are ordered in nondecreasing order.

³Horn, Johnson, *Topics in Matrix Analysis*, Cambridge, 1991

Weyl chamber

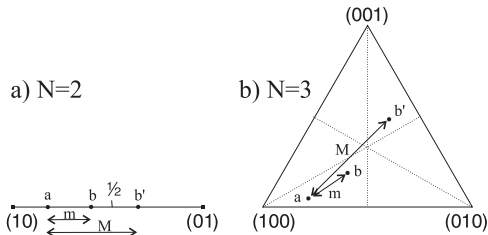
Maximal distance is achieved for points belonging to the opposite chambers

The minimal distance m between the uniray orbits of two quantum states are equal to the distances between the corresponding spectra a and b belonging to the same Weyl chamber shown for a) $N = 2$ and b) $N = 3$.

Weyl chamber

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Summary

- ▶ $\sum_{i=1}^n \sqrt{p_i^\uparrow q_i^\downarrow} \leq F(A, B) \leq \sum_{i=1}^n \sqrt{p_i^\uparrow q_i^\uparrow}$ which gives

$$\sum_{i=1}^n \sqrt{p_i^\uparrow q_i^\uparrow} \leq D_B(A, B) \leq \sum_{i=1}^n \sqrt{p_i^\uparrow q_i^\downarrow}$$

for Bures distance

- ▶ for trace distance we have

$$\sum_{i=1}^n |p_i^\uparrow - q_i^\uparrow| \leq \text{tr}|A - B| \leq \sum_{i=1}^n |p_i^\uparrow - q_i^\downarrow|$$

More details in arXiv:0711.4286

Final remarks

- ▶ Hilber-Schmidt distance does not provide good measure of distinguishability
- ▶ Bures distance and trace distance can determinate when the states can be deterministically discriminated
- ▶ looking for distinguishable states in a rotationally invariant subset it is sufficient to analyze the set of “classical” density matrices

Thank you!